A Simple Estimation of Parameters for Discrete Distributions from the Schröter Family

Friday I. Agu[1] | *Slovak Academy of Sciences, Bratislava, Slovakia*
Ján Mačutek[2] | *Constantine the Philosopher University in Nitra, Nitra, Slovakia*
Gábor Szűcs[3] | *Comenius University, Bratislava, Slovakia*

Received 12.9.2022, **Accepted** (reviewed) 3.10.2022, **Published** 16.6.2023

Abstract

One of the common challenges in actuarial mathematics is finding a model for the number of claims and claim severity. We focus on one of the suggested models, namely, on the Schröter family of discrete probability distributions. Furthermore, we introduce a simple and easy-to-compute parameter estimate for this family of distributions, which can be used especially as initial values in optimization algorithms that are needed to compute other estimates.

Keywords	***DOI***	***JEL code***
Discrete probability distributions, Schröter recursive formula, parameter estimation, aggregate claim distribution	*https://doi.org/10.54694/stat.2022.42*	*C46, C13, C25*

INTRODUCTION

In actuarial mathematics, there has been a significant active interest in modeling the number of claims and claim severities of a collective risk model for life and non-life insurance. A reliable model enables predictions that could help insurance companies set competitive prices for insurance portfolios and maintain adequate investment for the next operational year. Also, the insurance company could decide on the appropriate margins for the cost of each portfolio to account for future uncertainty.

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 81743 Bratislava, Slovakia. E-mail: agu1@uniba.sk. Also Department of Statistics, University of Calabar, Calabar, Nigeria.
[2] Department of Mathematics, Faculty of Natural Sciences and Informatics, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, 949 01 Nitra, Slovakia. E-mail: jmacutek@yahoo.com. Also Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia.
[3] Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina F1, 842 48 Bratislava, Slovakia.

A “naïve” approach to obtaining accurate aggregate claims is to find an appropriate family of counting distributions and to fit them to the number of claims and claim severities separately. However, this approach fails more often than not. The distribution of aggregate claims is, in fact, a convolution of the distributions of the number of claims and the claim severity.

Therefore, one of the common problems in actuarial mathematics is modeling the aggregate claims distribution of a collective risk model, with the claim severity and the number of claims considered as discrete random variables. In the literature, distributions such as compound negative binomial and compound Poisson have been studied extensively and used to model the aggregate claims distribution based on the convolution approach. Also, the theoretical aspects of compound distributions are in Johnson et al. (2005). Bening and Korolev (2012) focus on the compound Poisson distributions and their applications in actuarial and financial mathematics. Wimmer and Altmann (1999) enumerate several examples of these distributions.

From a theoretical perspective, the convolution approach is unambiguous. However, when the number of claim events increases, the computation of the aggregate claim distribution using the convolution approach becomes difficult. This problem has led researchers to search for alternative methods for computing the aggregate claim distribution of collective risk models. In this paper, we focus on one of them, namely, the recursive method.

The recursive method (Sundt and Vernic, 2009) assumes that the claim severity distribution is discrete and can compute aggregate claims recursively when the number of claims increases. It does not involve computing several convolutions of the conditional distribution of the number of claim events and requires far less computer time.

Panjer (1981) introduced the famous recursive formula:

$$P_n = \left(a + \frac{b}{n}\right) P_{n-1}, \quad n = 1, 2, \ldots \tag{1}$$

where a and b are parameters (by definition, henceforth $P_n = 0$ for $n < 0$). It addressed the computational problems of the convolution approach. However, only a few distributions – binomial, Poisson, and negative binomial[4] – satisfy the formula. Therefore, several suggestions to generalize (1) appeared in the literature, for example, see Schröter (1990), and Sundt (1992).

1 SCHRÖTER DISTRIBUTION FAMILY

Schröter’s (1990) second-order recursive formula is:

$$P_n = \left(a + \frac{b}{n}\right) P_{n-1} + \frac{c}{n} P_{n-2}, \quad n = 1, 2, 3, \ldots \tag{2}$$

where a, b, and c are parameters. It is obvious that (1) is a special case of (2) for $c = 0$. In addition to the distributions given by the Panjer’s recursion (1), the distribution family defined by (2) contains also convolutions of the Poisson distribution and another distribution from the Panjer family. Thus, the Schröter family is more flexible and can better capture the behavior of the number of claims and their severity. It can also be applied within actuarial reserving approaches, such as e.g. the one proposed by Maciak et al. (2021).

Also, Schröter (1990) derived some basic properties of distributions given by (2). These distributions have the probability generating function:[5]

[4] The Panjer’s Formula (1) defines, in fact, the Katz family of discrete probability distributions, see Wimmer and Altmann (1999).

[5] Formulas (3)–(6) are true if $a \neq 0$. However, they become more complicated if $a = 0$.

$$G(s)=e^{-\frac{c}{a}(s-1)}\left(\frac{1-a}{1-as}\right)^{\frac{a(a+b)+c}{a^2}}, \tag{3}$$

the explicit expression for the probability mass function:

$$P_n=e^{\frac{c}{a}}(1-a)^{\frac{a(a+b)+c}{a^2}}\sum_{j=0}^{n}\binom{\frac{a(a+b)+c}{a^2}+j-1}{j}\frac{\left(-\frac{c}{a}\right)^{n-j}a^j}{(n-j)!}, \tag{4}$$

the mean:

$$\mu=\frac{a+b+c}{1-a}, \tag{5}$$

and the variance:

$$\sigma^2=\frac{a+b+(2-a)c}{(1-a)^2}. \tag{6}$$

Several problems are related to the discrete probability distributions given by (2), which remain open. The parametric space of these distributions has not yet been specified. Although, Schröter (1990) presents some conditions that the parameters must satisfy. Another area that deserves a detailed investigation is parameter estimation.

2 PARAMETER ETIMATION

Due to a relatively complicated formula for the probability mass function (4), classical parameter estimation methods have no explicit results. For example, the maximum likelihood method results in a system of equations that has no explicit solutions and must be solved numerically. Luong and Garrido (1993) mention potential problems of such numerical solutions.

Therefore, Luong and Garrido (1993) suggested an estimation method specifically for recursively defined probability distributions, based on minimizing the quadratic distance (the recursion formula is considered a linear regression model). Luong and Doray (2002), and Doray and Haziza (2004) further elaborated on this idea. The minimum quadratic distance estimations have, under certain conditions, desirable properties (asymptotic normality, consistency, asymptotic efficiency). However, computations still require numerical methods and the use of software (they involve, e.g., matrix inversion).

We derive a simple, easy-to-compute estimation of parameters for distributions from the Schröter family. In Formulas (5) and (6), we replace the mean and variance with their empirical counterparts and the parameters with their estimates. That is, we have:

$$\bar{x}=\frac{\hat{a}+\hat{b}+\hat{c}}{1-\hat{a}}, \tag{7}$$

$$s^2=\frac{\hat{a}+\hat{b}+(2-\hat{a})\hat{c}}{(1-\hat{a})^2}. \tag{8}$$

Next, denote N the sample size, f_0, f_1, f_2, ... the observed frequencies of values 0, 1, 2, …, and k the number for which the sum of three neighbouring frequencies $f_k+f_{k+1}+f_{k+2}$ attains its maximum. The empirical analogue of (2) for $n=k$ is the equation:

$$\hat{p}_k = \left(\hat{a} + \frac{\hat{b}}{k}\right)\hat{p}_{k-1} + \frac{\hat{c}}{k}\hat{p}_{k-2}, \tag{9}$$

where $\hat{p}_k$, $\hat{p}_{k-1}$, and $\hat{p}_{k-2}$ are empirical probabilities (i.e. $\hat{p}_k = \frac{f_k}{N}$).[6]

The parameters will be estimated by solving the system of Formulas (7), (8), and (9). From (7) we have:

$$\bar{x}(1-\hat{a}) - \hat{c} = \hat{a} + \hat{b}. \tag{10}$$

Using (10), Formula (9) can be rewritten as:

$$k\hat{p}_k = \left[\hat{a}(k-1+1) + \hat{b}\right]\hat{p}_{k-1} + \hat{c}\hat{p}_{k-2} = \hat{a}\left(k-1-\bar{x}\right)\hat{p}_{k-1} + \bar{x}\,\hat{p}_{k-1} + \hat{c}\left(\hat{p}_{k-2} - \hat{p}_{k-1}\right), \tag{11}$$

and from (11) it follows that:

$$\hat{c} = \frac{k\hat{p}_k - \hat{a}\left(k-1-\bar{x}\right)\hat{p}_{k-1} - \bar{x}\,\hat{p}_{k-1}}{\hat{p}_{k-2} - \hat{p}_{k-1}}. \tag{12}$$

Combining (8) and (10) gives:

$$s^2\left(1-\hat{a}\right) = \bar{x} + \hat{c}, \tag{13}$$

and solving (12) and (13) yields the estimate:

$$\hat{a} = \frac{\left(s^2 - \bar{x}\right)\left(\hat{p}_{k-2} - \hat{p}_{k-1}\right) - k\hat{p}_k + \bar{x}\,\hat{p}_{k-1}}{s^2\left(\hat{p}_{k-2} - \hat{p}_{k-1}\right) - \left(k-1-\bar{x}\right)\hat{p}_{k-1}}. \tag{14}$$

Using (10), (13), and (14) we obtain estimates of the other two parameters:

$$\hat{c} = s^2\left(1-\hat{a}\right) - \bar{x}, \tag{15}$$

$$\hat{b} = \bar{x}(1-\hat{a}) - \hat{a} - \hat{c}. \tag{16}$$

3 NUMERICAL APPLICATIONS

3.1 Simulation study

We performed a simple simulation study to gain insight into the behavior of the estimates derived in Section 2.[7] We set parameter values as $a = 0.6$, $b = 2.6$, and $c = -1.1$ and generated 100 000 random numbers from this distribution. The generated numbers become the random sample from which we estimate the parameters. Furthermore, we repeat this procedure 10 000 times (that is, we obtained 10 000 estimates of each parameter). Table 1 and Figure 1 present the descriptive statistics of the estimated parameters.

[6] In principle, any trinity of neighbouring empirical probabilities can be used in Formula (9).

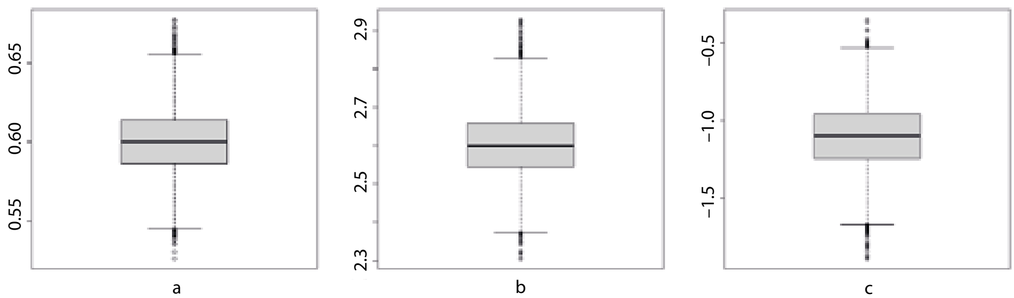
[7] All computations were performed in the statistical software environment R: <*www.r-project.org*>.

Table 1 The descriptive statistics of the estimates

	$\hat{a}$	$\hat{b}$	$\hat{c}$
Mean	0.60033	2.60136	–1.103495
Standard deviation	0.02063	0.08455	0.21174
Minimum	0.52549	2.30470	–1.89726
Maximum	0.67759	2.91813	–0.34615

Source: Own study

Figure 1 Boxplots of the estimates



Source: Own study

The means from Table 1 indicate that the estimates (at least for parameters a and b) could even be unbiased.[8] However, their standard deviations (especially for $\hat{c}$) seem not to converge to 0, leaving thus their consistency doubtful.

3.2 Car accident injuries

We applied the Schröter recursive relation (2) as a model for car accident injury data.[9] We consider the accidents that happened in the Olomouc region of the Czech Republic from January 1, 2021, to December 31, 2021, and all types of injuries (deadly, serious, and minor). In Table 2, we present the number of days with a particular number of injuries (for example, there were 40 days without an injury, 64 days with one injury, etc.).

Table 2 Car accident injuries in the Olomouc region from January 1, 2021, to December 31, 2021 (x – number of injuries, $f(x)$ – number of days with x injuries)

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0	40	3	55	6	29	9	7
1	64	4	33	7	22	10	5
2	60	5	39	8	8	12	3

Source: Own study, using surveys from <*www.irozhlas.cz/nehody*>

[8] The values from Table 1 are quite stable when simulations are rerun under the same conditions, with the means and the standard deviations changing at the third decimal place at most.

[9] The data were created from surveys (the webpage of the Czech public radio broadcaster, accessed on 10 October 2022) available at: <*www.irozhlas.cz/nehody*>.

The model achieves a good fit in terms of the Pearson chi-square test (p = 0.1677; $\hat{a}$ = 0.451, $\hat{b}$ = 1.127, $\hat{c}$ = 0.254). We note that this p-value was computed using parameter estimates given by Formulas (14)–(16), and it can be improved if, for example, we use the minimum chi-square method to estimate the parameters (with our estimations serving as initial values in optimization algorithms).

CONCLUSION

The paper presents new parameter estimations for discrete probability distributions from the Schröter family. They are very easy to compute, as they are given explicitly and do not involve iteration algorithms, numerical optimization etc., which is the case of previously suggested estimations (Luong and Garido, 1993; Luong and Doray, 2002; Doray and Haziza, 2004). The new Formulas (14)–(16) for parameter estimation can thus be used to gain a preliminary insight into data, and especially as initial values in numerical procedures which are used in the abovementioned more sophisticated approaches.

ACKNOWLEDGEMENT

Supported by research projects VEGA 2/0096/21 (F. I. Agu and J. Mačutek), APVV-21-0216 (J. Mačutek), and the Operational Programme Integrated Infrastructure (OPII) for the project 313011BWH2: "InoCHF – Research and development in the field of innovative technologies in the management of patients with CHF", co-financed by the European Regional Development Fund (J. Mačutek).

References

BENING, V. E., KOROLEV, V. Y. (2012). *Generalized Poisson Models and their Applications in Insurance and Finance*. Berlin, Boston: de Gruyter.

DORAY, L. G., HAZIZA, A. (2004). Minimum distance inference for Sundt's distributions. In: ANTOCH, J. (eds.) *COMPSTAT 2004 – Proceedings in Computational Statistics*, Heidelberg: Physica-Verlag.

JOHNSON, N. L., KEMP, A. W., KOTZ, S. (2005). *Univariate Discrete Distributions*. Hoboken: Wiley.

LUONG, A., DORAY, L. G. (2002). General quadratic distance methods for discrete distributions are definable recursively [online]. *Insurance: Mathematics and Economy*, 30(2): 255–257. <https://doi.org/10.1016/S0167-6687(02)00100–2>.

LUONG, A., GARRIDO, J. (1993). Minimum quadratic distance estimation for a parametric family of discrete distributions defines recursively [online]. *Australian Journal of Statistics*, 35(1): 59–67. <https://doi.org/10.1111/j.1467-842X.1993.tb01312.x>.

MACIAK, M., OKHRIN, O., PEŠTA, M. (2021). Infinitely stochastic micro reserving [online]. *Insurance: Mathematics and Economics*, 100: 30–58. <https://doi.org/10.1016/j.insmatheco.2021.04.007>.

PANJER, H. H. (1981). Recursive evaluation of a family of compound distributions [online]. *ASTIN Bulletin*, 12(1): 22–26. <https://doi.org/10.1017/S0515036100006796>.

SCHRÖTER, K. J. (1990). On a family of counting distributions and recursions for related distributions [online]. *Scandinavian Actuarial Journal*, 2–3: 161–175. <https://doi.org/10.1080/03461238.1990.10413879>.

SUNDT, B. (1992). On some extensions of Panjer's class of counting distributions [online]. *Scandinavian Actuarial Journal,* 1990 (2–3): 161–175. <https://doi.org/10.2143/AST.22.1.2005127>.

SUNDT, B., VERNIC, R. (2009). *Recursions for Convolutions and Compound Distributions with Insurance Applications*. Berlin, Heidelberg: Springer.

WIMMER, G., ALTMANN, G. (1999). *Thesaurus of Univariate Discrete Probability Distributions*. Essen: Stamm.