# Constructions of the Average Rate of Return of Pension or Investment Funds Based on Chain Indices

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***Abstract***

In this paper we consider the problem of the proper construction of the average rate of return of pension (or investment) funds. We refer to some economical postulates given by Gajek and Kaluszka (2000). We present, discuss and compare several measures of the average rate of return of funds. We also present alternative measures based on original chain indices. We take into consideration discrete and continuous time stochastic models.

| ***Keywords*** | ***JEL code*** |
|---|---|
| *Average rate of return, stochastic process, martingale, chain indices* | *C43, G12, G23* |

## INTRODUCTION

Open pension funds and investment funds are institutions that should invest their client's money in the most effective way. There is a number of measures for the efficiency of these investments (see Domański et al., 2011; Białek, 2008). The measures should be well defined – it means that all changes of fund's assets, connected with any investment, should have impact on the given measure. The information about the average return of the group of funds is very important both for fund clients and fund managers. Firstly, it allows to compare the result of the given fund to the rest of funds. It may be helpful to clients in making a decision about money allocation. Secondly, having the knowledge about the average returns of investment funds from different sectors (manufacturing, agricultural, service etc.) we have some information about the financial situation within these sectors. And finally, in the case of pension funds we can find law regulations defining the *minimal rate of return* of funds based on the average rate of return. For example, in the Polish law regulations (The Law on Organization and Operation of Pension Funds, Art. 173, Dziennik Ustaw Nr 139 poz. 934, Art. 173; for the English translation see Polish Pension…, 1997) the half of the average return of a group of funds or the averege return minus four percentage points (depending on which of these values is higher) determined (till February 2014) a minimal rate for any pension fund. In the case of deficit the weak fund had to cover it. It was always a very dangerous situation for this fund.[2] In the Polish law the following definition of the average return of a group of pen-

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[2] In Poland, in 2001 and 2002 Bankowy Fund did not reach the minimal rate of return.

sion funds could be found (only from 1st February 2014 the new law regulation has been in effect and according to which there is no need to calculate the minimal rate of return of funds and the average return - see Dziennik Ustaw 2013, poz. 1717):

$$\bar{r}_0(T_1,T_2)=\sum_{i=1}^{n}\frac{1}{2}r_i(T_1,T_2)\cdot\Big(\frac{A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_i(T_2)}{\sum_{i=1}^{n}A_i(T_2)}\Big), \tag{1}$$

where $r_i(T_1,T_2)$ denotes the rate of the $i$ – th fund during a given time period $[T_1,T_2]$ and $A_i(t)$ denotes the value of $i$ – th fund's assets at time $t$. Since 2004 till the February 2014 the results of funds for the last 36 months had been verified twice a year. Unfortunately, the measure defined in (1) does not satisfy some economic postulates given by Gajek and Kałuszka (2000). Moreover, considering an even number of funds, where half of them have the return rates equal to 50% and the rest of funds have the return rates equal to (–50%), we should get the real average return rate on the level 0%. But using formula (1) we get 12.5%. In our opinion, this in an argument for searching new definitions of the avarege rate of return of a group of funds.

## 1 POSTULATES FOR THE AVERAGE RATE OF RETURN

At the first sight the problem of constructing the avarege rate of return seems to be straightforward. But if we look at postulates coming from Gajek and Kałuszka (2000), which are quite natural and economical legitimate, we have to verify this opinion. Let us denote by $p_i(t)$ th value of the participation unit of the $i$ – th fund at time $t$, and $q_i(t)$ - the number of units of the $i$ – th fund at time $t$. Below we present and disccuss the postulates for the average rate of return of a group of funds ($\bar{r}(T_1,T_2)$).

**Postulate 1**

In the case when the group consists of one fund ($n = 1$) then:

$$\bar{r}(T_1,T_2)=r_1(T_1,T_2), \tag{2}$$

where: $$r_1(u,u+1)=\frac{p_1(u+1)-p_1(u)}{p_1(u)}. \tag{3}$$

**Postulate 2**

If all funds have the same values of their accounting units all the time, i.e.

$$p_i(t) = p_j(t),\ \text{for } i \neq j,\ t \in [T_1,T_2], \tag{4}$$

then it holds:

$$\bar{r}(T_1,T_2)=\frac{p_1(T_2)-p_1(T_1)}{p_1(T_1)}. \tag{5}$$

It means that if the unit's value changes in time in the same way in all funds then it does not matter if the clients allocate from one fund to another or where the newcomers place themselves; their individual return rates will always be the same.

**Postulate 3**

If the number of units of every fund is constant during the time interval $[T_1,T_2]$, then:

$$\bar{r}(T_1,T_2)=\frac{\sum_{i=1}^{n}A_i(T_2)-\sum_{i=1}^{n}A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}. \tag{6}$$

In fact, when none of the clients change the fund or come into or out of the business, then any change of assets of the $i$ – th fund reflects only the investment results of the $i$ – th fund. Moreover, postulate 3 implies also postulate 3', namely:

**Postulate 3'**

Under assumptions from postulate 3, if the initial assets at time $t = T_1$ of every fund are the same and for some k ≤ $n/2$ it holds $r_1 = -r_{k+1}$, $r_2 = -r_{k+2}$,..., $r_k = -r_{2k}$, $r_{2k+1} = 0$ ,..., $r_n = 0$ then

$$\bar{r}(T_1,T_2) = 0. \tag{7}$$

**Postulate 4**

For every $t \in [T_1,T_2]$ it shoul hold:

$$1 + \bar{r}(T_1,T_2) = [1 + \bar{r}(T_1, t)][1 + \bar{r}(t, T_2)]. \tag{8}$$

Postulate 4 is a multiplication rule that says that the average rate of return since $T_1$ until $T_2$ should equal the average return since $t$ until $T_2$, given the average return since $T_1$ until $t$. Let us notice that the individual rate of return defined in (3) satisfies postulate 4.

**Postulate 5**

Let us assume that $i$ – th fund obtains the highest return rates and the $k$ – th fund obtains the lowest raturn rates on each time interval $[t,t+1] \subseteq [T_1,T_2]$. Then we should observe:

$$r_k(T_1,T_2) \le \bar{r}(T_1,T_2) \le r_i(T_1,T_2). \tag{9}$$

Postulate 5 means that the average return rate is not greater than the rate corresponding to the case when all clients allocate at each $t \in [T_1,T_2]$ to the fund obtaining the highest return rate and not lower than the rate corresponding to the case in which all clients allocate to the fund obtaining the lowest return rate.

**Postulate 6**

If for some $k \in \{1,2,..., n\}$ it holds

$$\max_{i \neq k} A_i(t) \le \theta A_k(t) \text{ for any } t \in [T_1,T_2], \tag{10}$$

then we observe

$$\lim_{\theta \to 0} \bar{r}(T_1,T_2) = \frac{p_k(T_2) - p_k(T_1)}{p_k(T_1)}. \tag{11}$$

Postulate 6 means that the influence of small funds (with small asstets) on the average return is negligible.

**Postulate 7**[3]

If funds are grouped and if the average rate of return of each group is calculated over the time interval $[t,t+1] \subseteq [T_1,T_2]$, then the average rate of return of groups equals to the average rate of return of all funds over the the time interval .

[3] In the original paper of Gajek and Kałuszka (2002) authors treat the postulate 7 as one of properties of the proposed average rate of return. In our opinion it has an axiomatic character and should be treated as a postulate.

**Remark 1**

The above postulates decribe partly a kind of economical intuition and partly mathematical consistency of any good definition of a weighted average rate of return of a group of pension or investment funds. For example the Polish definition presented in (10) does not satisfy postulates 3, 3', 4 and 7 (the proof is easy and thus omitted, see the example 1). But the construction of a proper definition of the average rate of return is not obvious. For instance, even the well known and popular Value Line Composite Index[4] (VLIC index) defined as:

$$VLIC(T_1,T_2) = [(1+r_1(T_1,T_2))(1+r_2(T_1,T_2) \cdot \ldots \cdot (1+r_n(T_1,T_2))]^{\frac{1}{n}} - 1 = \Big[\prod_{i=1}^{n} \frac{p_i(T_2)}{p_i(T_1)}\Big]^{\frac{1}{n}} - 1, \quad (12)$$

does not satisfy postulates 3, 3', 6 and 7 (see the example 1). Let us also notice that the *VLIC* formula can be obtained as a value of the unweighted Jevons' index minus one.

**Example 1**

We show that measures defined in (1) and (12) does not satisfy postulate 7. Let us consider $n = 5$ funds with the same value of asstes during the time interval $[T_1,T_2]$ and their results as follows:

$r_1(T_1,T_2) = 0{,}05$, $r_2(T_1,T_2) = 0{,}07$, $r_3(T_1,T_2) = 0{,}12$, $r_4(T_1,T_2) = -0{,}03$, $r_5(T_1,T_2) = 0$.

We get for the whole group of funds:

$$\bar{r}_0(T_1,T_2) = \frac{1}{5}\sum_{i=1}^{5} r_i(T_1,T_2) = 0.042,$$

$$VLIC(T_1,T_2) = [\prod_{i=1}^{5}(1+r_i(T_1,T_2)]^{\frac{1}{5}} - 1 = 0.040.$$

Let us assume that funds 1 and 2 are in the first group (I), and funds 3, 4 and 5 are in the second group (II). After calculations we get the following results for groups:

$$\bar{r}_0^{I}(T_1,T_2) = \frac{1}{2}(r_1(T_1,T_2) + r_2(T_1,T_2)) = 0.06,$$

$$\bar{r}_0^{II}(T_1,T_2) = \frac{1}{3}(r_3(T_1,T_2) + r_4(T_1,T_2) + r_5(T_1,T_2)) = 0.03,$$

$$VLIC^{I}(T_1,T_2) = \sqrt{(1+r_1(T_1,T_2))(1+r_2(T_1,T_2))} - 1 = 0.060,$$

$$VLIC^{I}(T_1,T_2) = \sqrt[3]{(1+r_1(T_1,T_2))(1+r_2(T_1,T_2))(1+r_3(T_1,T_2))} - 1 = 0.028.$$

Now, let us calculate the average rate of return for joined groups:

$$\bar{r}_0^{I+II}(T_1,T_2) = \frac{1}{2}\cdot(\bar{r}_0^{I}(T_1,T_2)\cdot\frac{4}{5} + \bar{r}_0^{I}(T_1,T_2)\cdot\frac{6}{5}) = 0.040 \neq \bar{r}_0(T_1,T_2),$$

$$VLIC^{I+II}(T_1,T_2) = \sqrt{(1+VLIC^{I}(T_1,T_2))(1+VLIC^{II}(T_1,T_2))} - 1 = 0.043 \neq VLIC(T_1,T_2).$$

Thus, neither $\bar{r}_0$ nor *VLIC* satisfies the postulate 7.

4 This index containing approximately 1675 companies from the NYSE, American Stock Exchange, and Nasdaq.

In the next part of this paper we consider discrete and continuous time stochastic models and present several definitions of the average of return that fulfill postulates 1–7.

## 2 PROPOSITIONS OF THE AVERAGE RATE OF RETURN IN A DISCRETE TIME STOCHASTIC MODEL

### 2.1 Significations and assumptions

Let us consider a group of $n$ pension or investment funds that start their activity selling accounting units at the same price. We observe them in discrete time moments $\{t = 0,12,....\}$. Let us define a probability space $(\Omega, \Im, P)$. Let $F = \{\Im_t : t = 0,1,2,...\}$ be a filtration, i.e. each $\Im_t$ is an $\sigma$ – algebra of $\Omega$ with $\Im_0 \subseteq \Im_s \subseteq \Im_t \subseteq \Im$ for any $s < t$. Without loss of generality, we assume $\Im_0 = \{\emptyset, \Omega\}$. The filtration $F$ describes how the information about the market is revealed to the observer. We consider the following state-variables:

$p_i(t)$ – value of the participation unit of the $i$ – th fund at time $t$,

$q_i(t)$ – number of units of the $i$ – th fund at time $t$,

$A_i(t) = k_i(t)w_i(t)$ – value of $i$ – th fund's assets at time $t$,

$A(t) = \sum_{i=1}^{n} A_i(t)$,

$A^*_i(t) = A_i(t) / A(t)$ – the percentage of a relative value of assets of the $i$ – th fund at time $t$.

We assume that:

- All investments are infinitely divisible.
- There are no transaction costs or taxes and the assets pay no dividends.
- Member does not pay for allocation of his/her wealth.
- There is no consumption of funds.

The presented, technical assumptions make the mathematical transformations easier but the assumptions do not influence the general character of the discussion. The presented research on real data shows that there are still some benefits of using the proposed measures although some of the assumptions can not be satisfied (for example a member can pay for allocation of his/her wealth). Thus the properties of the discussed measures do not depend on the above assumptions.

Here and subsequently, the symbol $X = Y$ means that the random variables $X,Y$ are defined on $(\Omega, \Im, P)$ and $P(X = Y) = 1$. We assume that each $p_i(t)$ and $q_i(t)$ is adapted to $F = \{\Im_t : t = 0,1,2,...\}$ which means that each $p_i(t)$ and $q_i(t)$ is measurable with respect to $\Im_t$. Next we consider some time interval of observations given by $[T_1,T_2]$.

### 2.2 The measure of Gajek and Kałuszka and its connection with chain indices

Under the above assumptions and symbols Gajek and Kałuszka (2001) proposed the following definition of the average rate of return of a group of funds:

$$\bar{r}_{GK}(T_1,T_2) = \prod_{t=T_1}^{T_2-1}(1+\sum_{i=1}^{n} A_i^*(t) r_i(t,t+1)) - 1. \tag{13}$$

The definition (13) satisfies all the economic postulates 1–7 (see Gajek, Kałuszka, 2001). In the mentioned paper the authors proved also the following theorems.

**Theorem 1**

If the number of units of each of the fund is constant on the time interval then we have:

$$\bar{r}_{GK}(t,t+1) \le \bar{r}_0(t,t+1) \tag{14}$$

and in the natural case of:

$$\exists i, j \; \frac{p_i(t+1)}{p_i(t)} \neq \frac{p_j(t+1)}{p_j(t)}, \tag{15}$$

we obtain:

$$\bar{r}_{GK}(t,t+1) < \bar{r}_0(t,t+1). \tag{16}$$

The inequality (16) suggests that the average return defined in the Polish law overestimates the real average rate of return of a group of funds.

**Theorem 2**

If $\{p_i(t) : t = 0,1,2,....\}$ is an $F$ – martingale[5] for each $i$, then $\{\bar{r}_{GK}(0,t) : t = 0,1,2,...\}$ is also an $F$ – martingale. Moreover, in case when $\{p_i(t) : t = 0,1,2,....\}$ is an $F$ – submartingale (resp. $F$ – supermartingale) for each $i$, then $\{\bar{r}_{GK}(0,t) : t = 0,1,2,...\}$ is an $F$ – submartingale (resp. $F$ – supermartingale).

**Remark 2**

The average rate of return defined in the Polish law ($\bar{r}_0$) in general is not a martingale provided the values of units are martingales (see Gajek and Kałuszka, 2001).

In this part of the paper we treat the group of fund as some aggregate that contains $n$ commodities (funds) with prices $p_i(t)$ and quantities $q_i(t)$, where $t \in [T_1, T_2]$. Let us denote by $P^L(t,t+1)$ the Laspeyres price index defined as follows (see von der Lippe, 2007):

$$P^L(t,t+1) = \frac{\sum_{i=1}^{n} q_i(t)p_i(t+1)}{\sum_{i=1}^{n} q_i(t)p_i(t)}. \tag{17}$$

Let us notice that the definition (13) can be written with the use of the Laspeyres chain index $\overline{P}^{LC}$. In fact we have (see Białek, 2011):

$$\begin{aligned}
\overline{P}^{LC}(T_1,T_2) - 1 &= \prod_{t=T_1}^{T_2-1} P^L(t,t+1) - 1 = \prod_{t=T_1}^{T_2-1} \frac{\sum_{i=1}^{n} q_i(t)p_i(t+1)}{\sum_{i=1}^{n} q_i(t)p_i(t)} - 1 = \\
&= \prod_{t=T_1}^{T_2-1} \Big( \sum_{i=1}^{n} \frac{q_i(t)p_i(t)}{\sum_{i=1}^{n} q_i(t)p_i(t)} \cdot \frac{p_i(t+1)}{p_i(t)} \Big) - 1 = \prod_{t=T_1}^{T_2-1} \Big( 1 + \sum_{i=1}^{n} \frac{q_i(t)p_i(t)}{\sum_{i=1}^{n} q_i(t)p_i(t)} \cdot \frac{p_i(t+1) - p_i(t)}{p_i(t)} \Big) - 1 = \\
&= \prod_{t=T_1}^{T_2-1} (1 + \sum_{i=1}^{n} A_i^*(t) r_i(t,t+1)) - 1 = \bar{r}_{GK}(T_1,T_2).
\end{aligned} \tag{18}$$

The question is whether we can use another chain indices to obtain the well-constructed average rate of return of funds. The answer is positive and we present such definitions in the next part of this paper.

[5] It implies that $E(p_i(t)) = const$, where $E(X)$ means the expected value of random variable $X$.

### 2.3 A general formula of the average return rate and its special cases

According to presented postulates it can be shown that the proper definition of the average rate of return of funds can be written as some chain price index minus one, namely:

$$\bar{r}(T_1,T_2)=\prod_{t=T_1}^{T_2-1}P(t,t+1)-1\,, \tag{19}$$

where the general form of the price index $P(t,t+1)$ is as follows:

$$P(t,t+1)=\prod_{i=1}^{n}\left(\frac{p_i(t+1)}{p_i(t)}\right)^{w_i(A_i^*(t),A_i^*(t+1))}\,. \tag{20}$$

The weights $w_i$ used in (20) are positive and sum up to one since

$$w_i(A_i^*(t),A_i^*(t+1))=\frac{M(A_i^*(t),A_i^*(t+1))}{\sum_{i=1}^{n}M(A_i^*(t),A_i^*(t+1))}\,, \tag{21}$$

where $M\,(x, y)$ is some type of (weighted) mean of variables $x$ and $y$ (arithmetic, geometric, exponential, etc.).

**Remark 3**

Let us assume that $M\,(A^*_i\,(t),\ A^*_i\,(t+1))=A^*_i\,(t)$. Then from (19) and (20) we obtain:

$$\bar{r}(T_1,T_2)=\prod_{t=T_1}^{T_2-1}\prod_{i=1}^{n}\left(\frac{p_i(t+1)}{p_i(t)}\right)^{A_i^*(t)}-1=\prod_{t=T_1}^{T_2-1}\exp\left(\sum_{i=1}^{n}A_i^*(t)\ln\frac{p_i(t+1)}{p_i(t)}\right)-1=\bar{r}_B(T_1,T_2)\,, \tag{22}$$

where $\bar{r}_B(T_1,T_2)$ means the avarege rate of return proposed and discussed in the paper of Białek (2008). Let us notice that in this case the $P(t,t+1)$ formula is a logarithmic Laspeyres price index (see von der Lippe, 2007). Taking $M\,(A^*_i\,(t),\ A^*_i\,(t+1))=A^*_i\,(t)$ we get the measure $\bar{r}_{LP}(T_1,T_2)$ based on the logarithmic Paasche price index (see von der Lippe (2007)). If we assume $M\,(A^*_i\,(t),\ A^*_i\,(t+1))=A^*_i\,(t)+A^*_i\,(t+1))/2$ then we can express the average rate of return by the Törnqvist chain price index, namely we obtain:

$$\bar{r}(T_1,T_2)=\bar{r}_T(T_1,T_2)=\prod_{t=T_1}^{T_2-1}P^T(t,t+1)-1\,, \tag{23}$$

where Törnqvist price index is defined for moments (as basis) and as follows (see Balk and Diewert, 2001):

$$P^T(t,t+1)=\prod_{i=1}^{n}\left(\frac{p_i(t+1)}{p_i(t)}\right)^{\frac{1}{2}(A_i^*(t)+A_i^*(t+1))}\,. \tag{24}$$

**Remark 4 (The next step of generalization)**

Let us define for any $x, y \in [0,1]$

$$A_i^x(t)=\frac{p_i(t)q_i^{1-x}(t)q_i^x(t+1)}{\sum_{i=1}^{n}p_i(t)q_i^{1-x}(t)q_i^x(t+1)}\,, \tag{25}$$

$$A_i^y(t+1) = \frac{p_i(t+1)q_i^{1-y}(t)q_i^y(t+1)}{\sum_{i=1}^{n} p_i(t+1)q_i^{1-y}(t)q_i^y(t+1)} \,. \tag{26}$$

Let us assume that $\tilde{M}(x, y)$ denotes the logaritmic mean defined for positive arguments as follows:

$$\widetilde{M}(x,y) = \frac{x-y}{\ln x - \ln y} \,, \tag{27}$$

if $x \neq y$, and $\tilde{M}(x, y) = x$ if $x = y$ (see Carlson, 1972).

Let us define the geo-logarithmic family as the class of price indices $P_{xy}$ defined by (see Fattore, 2010):

$$P_{xy}(t,t+1) = \prod_{i=1}^{n} \left(\frac{p_i(t+1)}{p_i(t)}\right)^{\widetilde{w}_i(A_i^x(t),A_i^y(t+1))} , \tag{28}$$

where:

$$\widetilde{w}_i(A_i^x(t), A_i^y(t+1)) = \frac{\widetilde{M}(A_i^x(t), A_i^y(t+1))}{\sum_{i=1}^{n} \widetilde{M}(A_i^x(t), A_i^y(t+1))} \,. \tag{29}$$

From the axiomatic point of view the general formula (28) is well-constructed. Geo-logarithmic price indices satisfy for example the proportionality, the commensurability or the homogeneity (see Fattore, 2010). In the mentioned paper the author proves that an element of the $P_{xy}$ family is monotonic if and only if $x = y$. It is very interesting that in this case, when just $x = y$, we obtain (see Martini, 1992):

$$P_{xx}(t,t+1) = \frac{\sum_{i=1}^{n} p_i(t+1)q_i^{1-x}(t)q_i^x(t+1)}{\sum_{i=1}^{n} p_i(t)q_i^{1-x}(t)q_i^x(t+1)} \,. \tag{30}$$

Let us notice that the formula (28) corresponds to the formula (20). In a similar way to (19) we define:

$$\bar{r}_{xy}(T_1,T_2) = \prod_{t=T_1}^{T_2-1} P_{xy}(t,t+1) - 1 = \prod_{t=T_1}^{T_2-1} \prod_{i=1}^{n} \left(\frac{p_i(t+1)}{p_i(t)}\right)^{\widetilde{w}_i(A_i^x(t),A_i^y(t+1))} - 1. \tag{31}$$

It is an interesting, general formula of the average rate of return of funds. Let us notice that from (30) we get that $P_{00}$ is the Laspeyres price index, $P_{11}$ is the Paasche price index and $P_{0.50.5}$ is the Walsh price index (see Białek, 2012). Thus, the $\bar{r}_{00}$ measure is based on the Laspeyres chain index, the $\bar{r}_{11}$ formula is based on the Paasche chain index and the $\bar{r}_{0.50.5}$ measure is based on the Walsh chain index. Let us denote two last formulas by $\bar{r}_P(T_1,T_2)$ and $\bar{r}_W(T_1,T_2)$. The formula $\bar{r}_{00}$ does not need any additional signification since we have:

$$\bar{r}_{00}(T_1,T_2) = \prod_{t=T_1}^{T_2-1} P^L(t,t+1) - 1 = \bar{r}_{GK}(T_1,T_2) \,. \tag{32}$$

### 2.4 Comparison of measures $\bar{r}_0$, $\bar{r}_{GK}$ and $\bar{r}_B$

As we know, the process $\{\bar{r}_{GK}(0,t) : t = 0,1,2,...\}$ is a $F$ – martingale provided the processes of prices are also martingales (see Theorem 2). As it was mentioned, the Polish formula $\bar{r}_0(0,t)$ in general does not have this property. In fact, let us consider a group that consists of only $n = 2$ funds. Let us assume $q_1(t) = q_2(t) = q$ and $p_1(0) = p_2(0) = 1$. From (1) we have:

$$\bar{r}_0(0,t)=\sum_{i=1}^{2}\frac{1}{2}r_i(0,t)\cdot\Big(\frac{A_i(0)}{\sum_{i=1}^{n}A_i(0)}+\frac{A_i(t)}{\sum_{i=1}^{n}A_i(t)}\Big)=\sum_{i=1}^{2}\frac{1}{2}(p_i(t)-1)\Big(\frac{1}{2}+\frac{p_i(t)}{p_1(t)+p_2(t)}\Big)=$$

$$=\frac{1}{2}\Big(\frac{1}{2}p_1(t)+\frac{p_1^2(t)}{p_1(t)+p_2(t)}-\frac{p_1(t)}{p_1(t)+p_2(t)}+\frac{1}{2}p_2(t)+\frac{p_2^2(t)}{p_1(t)+p_2(t)}-\frac{p_2(t)}{p_1(t)+p_2(t)}-1\Big)= \tag{33}$$

$$=\frac{1}{4}(p_1(t)+p_2(t))+\frac{1}{2}\frac{p_1^2(t)+p_2^2(t)}{p_1(t)+p_2(t)}-1$$

Let as assume naturally that $P(p_1(t)=p_2(t))<1$ for any $t>0$, which leads to

$$(p_1(t)-p_2(t))^2>0, \tag{34}$$

and equivalently[6]

$$2(p_1^2(t)+p_2^2(t))>p_1(t)+p_2(t))^2. \tag{35}$$

From (33) and (35) we get:

$$\begin{aligned}E(\bar{r}_0(0,t)&=\frac{1}{4}(E(p_1(t))+E(p_2(t)))+\frac{1}{2}E\Big(\frac{p_1^2(t)+p_2^2(t)}{p_1(t)+p_2(t)}\Big)-1>\\&>\frac{1}{4}E(p_1(t))+\frac{1}{4}E(p_2(t))+\frac{1}{4}E\Big[\frac{(p_1(t)+p_2(t))^2}{p_1(t)+p_2(t)}\Big]-1=\frac{1}{2}E(p_1(t))+\frac{1}{2}E(p_2(t))-1.\end{aligned} \tag{36}$$

Let us notice that in this case, even if $p_1(t)$ and $p_2(t)$ are martingales the average of rate of return is not a martingale. In fact, then we have $E(p_i(t))=E(p_i(0))=1$, but from (36) we obtain:

$$E(\bar{r}_0(0,t))>0=E(\bar{r}_0(0,0)), \tag{37}$$

which confirms that the process $\{\bar{r}_0(0,t):t=0,1,2,...\}$ can not be a martingale (its expected value is not constant in time). The next theorem gives us a condition that allows us to treat the stochastic process $\{\bar{r}_B(0,t):t=0,1,2,...\}$ as a martingale (see Białek, 2005).

**Theorem 3**

If $\{p_i(0,t):t=0,1,2,...\}$ is a $F-$ martingale, for each $i$ and with the probability one we have:

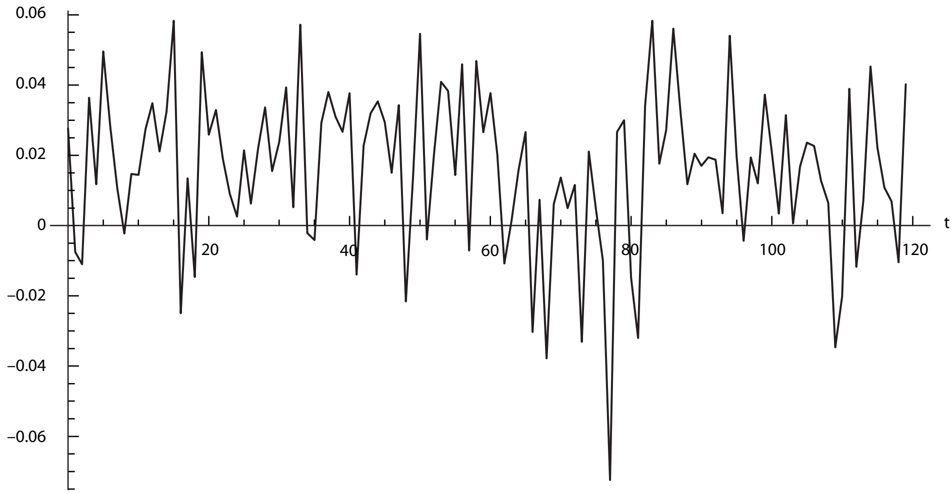
$$\lambda(t,t+1)\overset{def}{=}\sum_{i=1}^{n}A_i^*(t)\ln\frac{p_i(t+1)}{p_i(t)}\geq 0,\ \text{for any } t, \tag{38}$$

then $\{\bar{r}_B(0,t):t=0,1,2,...\}$ is also a $F-$ martingale.

The assumption (38) means that, in general, taking into consideration time intevals we observe (within the group of funds) more rises in prices than drops. Thus, during the financial crisis the process may not be a martingale since then it is difficult to fulfill (38). However, in the time of prosperity, as a rule the assumption (38) is satisfied (see Figure 1).

[6] This is a special case of the known enaquality: $(x_1+x_2+...+x_n)^2\leq n(x_1^2+x_2^2+...+x_n^2)$.

**Firure 1** Function $\lambda(t,t+1)$ for the case of group of open pension funds in Poland and time interval 06/2002–
–06/2012 *)




*) We consider monthly data and the financial crisis in Poland was (approximately) the strongest for $t \in [62, 83]$.
**Source:** Own calculations in Mathematica 6.0

The theorem 4 shows some relation between the discussed measures (see Białek, 2005).

**Theorem 4**
With probability one we have:

$$\bar{r}_B(T_1, T_2) \leq \bar{r}_{GK}(T_1, T_2), \tag{39}$$

and if $p_i(t+1) \approx p_i(t)$ for each $i$ and $t \in [T_1, T_2]$, then $\bar{r}_B(T_1, T_2) \approx \bar{r}_{GK}(T_1, T_2)$.

**Remark 5**
Let us define two random variables $\hat{P}$ and $\hat{Q}$ as follows:

$$\hat{P} = \frac{p_J(t+1)}{p_J(t)}, \tag{40}$$

$$\hat{Q} = \frac{q_J(t+1)}{q_J(t)}, \tag{41}$$

where $J$ is a random variable with distribution

$$P(J=j) = A_i^*(t), j = 1,2,..., n. \tag{42}$$

In the paper of Gajek and Kałuszka (2002) authors prove that:

$$\bar{r}_0(t,t+1) - \bar{r}_{GK}(t,t+1) = \frac{Cov(\hat{P}\hat{Q}, \hat{P})}{2E(\hat{P}\hat{Q})}. \tag{43}$$

Let us notice that in the case $P(\tilde{Q} = const) = 1$ we obtain from (43)

$$\bar{r}_0(t,t+1) - \bar{r}_{GK}(t,t+1) = \frac{Var(\hat{P})}{2(1+\bar{r}_{GK}(t,t+1))} \geq 0. \tag{44}$$

From (44) and Theorem 4 we get the following conclusion:

$$\bar{r}_0(t,t+1) \geq \bar{r}_{GK}(t,t+1) \geq \bar{r}_B(t,t+1), \tag{45}$$

and if $p_i(t+1) \approx p_i(t)$ that means $Var(\hat{P}) \approx 0$ we get:

$$\bar{r}_0(t,t+1) \approx \bar{r}_{GK}(t,t+1) \approx \bar{r}_B(t,t+1). \tag{46}$$

The assumption $P(\hat{Q} = const) = 1$ seems to be rather unnatural. In practice, the relatve increment of the number of units of each fund should be proportional to the relative increment of the value of unit, i.e.

$$(\hat{Q} = f(\hat{P}), \tag{47}$$

where $f: R_+ \rightarrow R_+$ is some nondecreasing function.

In the case of (47) we have (see Gajek and Kałuszka, 2002):

$$\bar{r}_0(t,t+1) - \bar{r}_{GK}(t,t+1) = \frac{E(\hat{P}^2 f(\hat{P})) - E(\hat{P}f(\hat{P}))E(\hat{P})}{E(\hat{P}f(\hat{P}))}. \tag{48}$$

From the following inequality for nondecreasing functions (see Mitrinovic et al., 1993):

$$E(\hat{P}^2 f(\hat{P})) \geq E(\hat{P}f(\hat{P}))E(\hat{P}), \tag{49}$$

from (48) we obtain again:

$$\bar{r}_0(t,t+1) - \bar{r}_{GK}(t,t+1) \geq 0. \tag{50}$$

Thus the formula $\bar{r}_0$ seems to overestimate the real value of the average rate of return.

## 2.5 Empirical study

Let us consider a group of $n = 14$ Polish open pension funds[7] and time interval of their observations: 06/2002–06/2012. Having monthly data on their numbers of clients and prices of units ($N = 120$ observations) we calculate the discussed measures of the average rate of return for several time intervals from the given period. Our results are presented in Table 1.

**Table 1** Considered average rates of return for some time intervals from the period 06/2002–06/2012

| Time interval | Measure of the average rate of return [%] | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\bar{r}_0(1,12)$ | $\bar{r}_{GK}(1,12)$ | $\bar{r}_B(1,12)$ | $\bar{r}_p(1,12)$ | $\bar{r}_{LP}(1,12)$ | $\bar{r}_T(1,12)$ | $\bar{r}_W(1,12)$ |
| [1, 24] | 24.38 | 24.33 | 24.32 | 24.33 | 24.35 | 24.33 | 24.33 |
| [1, 48] | 63.42 | 63.32 | 63.29 | 63.33 | 63.36 | 63.33 | 63.33 |
| [1, 72] | 83.14 | 82.96 | 82.91 | 82.96 | 83.00 | 82.95 | 82.95 |
| [1, 120] | 103.51 | 103.25 | 103.17 | 103.24 | 103.32 | 103.25 | 103.25 |
| [30, 90] | 41.41 | 41.41 | 41.38 | 41.40 | 41.43 | 41.41 | 41.41 |
| [60, 120] | 6.45 | 6.37 | 6.35 | 6.37 | 6.38 | 6.37 | 6.37 |

**Source:** Own calculations in Mathematica 6.0 based on data from <www.parkiet.pl>.

7 Here is the list of open pension funds in Poland in 2012: AIG, Allianz, Bankowy, Aviva, AXA, WARTA, AEGON, Generali, ING, Pekao, Pocztylion, Polsat, PZU, Nordea..

As we can notice, as a rule the Polish measure $\bar{r}_0$ has the highest value (the case of time interval [30, 90] is an exception) and the measure $\bar{r}_B$ has the smallest value (see also the simulation study). This observation seems to confirm the thesis of Theorem 4 and the conclusion from Remark 5. In fact, the Polish formula seems to overestimate the real value of the average rate of return. As it was mentioned, in the Polish law regulations the half of the average return of a group of funds or the average return minus four percentage points (depending on which of these values is higher) determines a minimal rate for any pension fund. In the case of deficit the *weak* pension fund has to cover it and thus it is always a very dangerous situation for this fund. Thus, from the funds' point of view, the definition $\bar{r}_B$ is "the safest". Nevertheless, there is a little difference in values of discussed measures in our research. It is easy to explain this fact because Polish pension funds invest in a very similar way. In other words, the criterion of the mimimal rate of return does not motivate funds to invest more efficiently and thus, funds have very similar portfolios. In such a situation the presented measures of the average return approximate each other (see Postulate 2).

### 2.6 Simulation study

Let us take into consideration a group of $n = 6$ funds observed at moments $t = 1,2,...,12$ and the following prices of units and numbers of units processes:

$$p_i(t) \sim N(\mu_i(t), \sigma_i(t)),\ i = 1,2,...,6,$$

$$q_i(t) \sim N(\hat{\mu}_i(t), \hat{\sigma}_i(t)),\ i = 1,2,...,6,$$

where $X \sim N(\mu, \sigma)$ denotes a random variable $X$ with a normal (Gaussian) distribution with a mean $\mu$ and a standard deviation $\sigma$. In our experiment we consider the following functions:

$$\mu_1(t) = 10 + 0.5t,\ \sigma_1(t) = 3,\ \hat{\mu}_1(t) = 200 + 5t,\ \hat{\sigma}_1(t) = 30,$$
$$\mu_2(t) = 100,\ \sigma_2(t) = 10 + t,\ \hat{\mu}_2(t) = 10 - 0.05t,\ \hat{\sigma}_2(t) = 1,$$
$$\mu_3(t) = 20,\ \sigma_3(t) = 2,\ \hat{\mu}_3(t) = 1000 + 50t,\ \hat{\sigma}_3(t) = 30 + 10t,$$
$$\mu_4(t) = 200 - t,\ \sigma_4(t) = 10 + t,\ \hat{\mu}_4(t) = 500 + 5t,\ \hat{\sigma}_4(t) = 30 - t,$$
$$\mu_5(t) = 10,\ \sigma_5(t) = 1 + 0{,}01t,\ \hat{\mu}_5(t) = 100,\ \hat{\sigma}_5(t) = 15,$$
$$\mu_6(t) = 50 + 3t,\ \sigma_4(t) = 4,\ \hat{\mu}_6(t) = 500,\ \hat{\sigma}_4(t) = 30 + 2t\ .$$

After calculations for $k = 10\ 000$ realizations of prices and numbers of units processes we get results presented in Table 2.

**Table 2** Basic characteristics of the considered average rates of return for the time interval [1, 12]

| Parameter | Measure of rate of return [%] | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\bar{r}_0(1,12)$ | $\bar{r}_{GK}(1,12)$ | $\bar{r}_B(1,12)$ | $\bar{r}_p(1,12)$ | $\bar{r}_{LP}(1,12)$ | $\bar{r}_T(1,12)$ | $\bar{r}_W(1,12)$ |
| mean | 11.00 | 7.41 | 1.50 | 7.30 | 13.70 | 7.30 | 7.39 |
| standard deviation | 8.20 | 8.79 | 8.70 | 8.80 | 9.50 | 8.80 | 8.80 |
| median | 10.70 | 7.20 | 1.50 | 7.00 | 13.40 | 7.30 | 7.40 |
| median deviation | 5.40 | 5.63 | 5.90 | 5.70 | 6.09 | 5.60 | 5.60 |
| minimum value | −10.08 | −16.40 | −22.51 | −15.63 | −11.82 | −15.80 | −15.80 |
| maximum value | 39.32 | 35.52 | 30.30 | 37.90 | 49.79 | 36.70 | 36.70 |

**Source:** Own calculations in Mathematica 6.0

As we can notice, the volatilities of all considered measures seem to be similar but there are significant differences between means and medians of some rates of return. Although $\bar{r}_{GK}$, $\bar{r}_P$, $\bar{r}_T$ and $\bar{r}_W$ have very close means and medians, the rest of average rates of return exhibit outstanding values of these parameters. For instance, according to the Theorem 4 and Remark 5 we can notice that the formula $\bar{r}_B$ has the value of mean and median smaller than $\bar{r}_{GK}$ and $\bar{r}_0$. Moreover, the mean and median of $\bar{r}_B$ are the smallest at all. It is also interesting, that the Polish measure $\bar{r}_0$ and the rate $\bar{r}_{LP}$ generates the highest values of mean, median and maximum. We obtain similar conclusions even if we consider a small time interval of observations (see Table 3). Thus in practice, it is very important which measures we use for calculations.

**Table 3** Basic characteristics of the considered average rates of return for the time interval [1, 3]

| Parameter | Measure of rate of return | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\bar{r}_0(1,3)$ | $\bar{r}_{GK}(1,3)$ | $\bar{r}_B(1,3)$ | $\bar{r}_p(1,3)$ | $\bar{r}_{LP}(1,3)$ | $\bar{r}_T(1,3)$ | $\bar{r}_W(1,3)$ |
| mean | 2.10 | 1.50 | 0.60 | 1.50 | 2.40 | 1.50 | 1.50 |
| standard deviation | 6.30 | 6.20 | 6.20 | 6.30 | 6.30 | 6.20 | 6.30 |
| median | 2.21 | 1.70 | 1.01 | 1.70 | 2.59 | 1.80 | 1.70 |
| median deviation | 4.30 | 4.30 | 4.30 | 4.30 | 4.31 | 4.30 | 4.30 |
| minimum value | –24.10 | –25.00 | –26.10 | –24.80 | –23.50 | –24.80 | –24.90 |
| maximum value | 20.00 | 19.90 | 19.50 | 19.80 | 21.10 | 19.90 | 19.80 |

**Source:** Own calculations in Mathematica 6.0

## CONCLUSIONS

The Polish definition of the average rate of return of a group of funds does not satisfy some economic postulates given by Gajek and Kałuszka (2000) although it had been in use in Poland for many years. Moreover this measure seems to overestimate the real value of the average return of funds. If funds invest similarly it does not matter which measure we use to calculate the average return of a whole group of funds. In another case the choice of the formula of the average rate of return is significant and important. We observe that the value of the formula $\bar{r}_B$ is the lowest and $\bar{r}_0$ and $\bar{r}_{LP}$ generate the highest values during the considered time interval (see Table 1, Table 2, and Table 3).

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