Modelling Marital Reverse Annuity Contract in a Stochastic Economic Environment

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Abstract

In the paper, we present the methodology of calculating the benefit of a marriage reverse annuity using the multiple state model for marriage life insurance. We model the probabilistic structure and cash flows arising from marriage reverse annuity contracts in the case of the joint-life status and the last surviving status assuming that the spouses' future lifetimes are independent. Usually, it is assumed that the interest rate is constant and the same through the years. It is not a realistic assumption. Therefore, this article's purpose is to calculate benefits under the assumption that the interest rate is a stochastic process or a fuzzy number model of the constant interest rate. We conduct a comparative analysis of the amount of benefit (taking into account the different frequency of their payment) for the different models of interest rates.

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Stochastic interest rate, fuzzy interest rate, reverse annuity contract, the joint-life status, the last surviving status	*https://doi.org/10.54694/stat.2022.2*	*C69, E47, G17, G22*

INTRODUCTION

This article aims to apply multiple state models for marriage insurances to model the marriage reverse annuity contract. A reverse annuity contract is one type of equity release contract. It is a sales model. The real estate owner receives a monthly whole life annuity in exchange for selling the real estate to a company (usually a mortgage fund) interested in buying it. The beneficiary has the right to live in the property until his death. In Poland, the reverse annuity contract has only been sold in individual form since 2005. This paper aims to analyze the benefits of a marriage reverse annuity contract in the case of last survivor status and joint living status.

Usually, in the actuarial literature, the interest rate is assumed constant and the same through the years. It is not a realistic assumption. Therefore, this article's purpose is to calculate reverse annuity

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benefits under the assumption that the interest rate is a stochastic process or a fuzzy number model of the constant interest rate. Many different interest rate models exist. The purpose of the article is to calculate net benefits (net cash flows). Hence the technical interest rate is applied. The expected future net present value benefits determined under the equivalence principle must cover the expected present value contributions at the time of entering into the contract. Therefore, inflation is not taken into account in this case. However, with rising inflation, the technical interest rate must be estimated appropriately since a reverse annuity contract is long-term.

We distinguish two ways of interest rate modelling, i.e., actuarial (the technical interest rate) and financial (the short-term rate). The following models of the stochastic interest rate are considered: the first-order autoregression, the Wiener process, the Vasicek and Cox-Ingersoll-Ross model. The third possibility considers the constant interest rate modelled by a fuzzy number. The fuzzy rate has not yet been applied to the determination of reverse annuity benefits. Therefore, the research hypothesis is that interest rates and their models (types) have a significant impact on the mortgage annuity benefit. In addition, the benefit is influenced by the frequency of payments. We show this effect.

The first section presents a literature review. Section 2 focuses on a discrete-time model, where the annuity is paid at the beginning of particular time units. We assume that the time-nonhomogeneous Markov chain describes the evolution of the contracted risk. Moreover, actuarial values are considered under the assumption of different types of interest rates. We propose to employ a matrix notation that makes calculations easier and provides a compact form for reverse annuity benefit formulas. This section is also dedicated to the characteristics of the interest rates models. Section 3 presents numerical examples and conclusions. The introduced matrix notation efficiently analyzes the influence of interest rate on the annuity installment. The numerical results are based on simulated data and the Polish Life Table, assuming that the spouses' future lifetimes are independent random variables. Section 4 presents discussion.

1 LITERATURE SURVEY

Multiple state models (MSM) have a wide application for describing a different kind of problems in finance and insurance, in particular analyzing cash flows arising from different kind of contracts. There is a vast literature on theoretical aspects and applications of MSM. Of particular note are the monographs e.g. (Cook and Lawless, 2018; Haberman and Pitacco, 2018; Hougaard, 2000; Huzurbazar, 2019) which show the potential of using this type of modelling. The general methodology of modern life insurance mathematics in the framework of a MSM can be found among others in (Bowers et al., 1986; Dębicka, 2012; Dickson et al., 2019; Norberg, 2002; Spierdijk and Koning, 2011). In particular MSM was used for analysis mortgage and reverse loan contracts e.g. (Dębicka et al., 2020; Dębicka and Marciniuk, 2014; Marciniuk, 2017; Marciniuk et al., 2020a; Zmyślona and Marciniuk, 2020).

The financial institutions in different countries propose the so-called equity release products for the retired (Hanewald et al., 2016), which provide an additional income to surrender their real estate. Various such products are available in the biggest world in the United States of America and Australia. The United Kingdom market is the largest European market for equity release contracts (Shao et al., 2015). Still, these contracts exist also in many other European countries (e.g. Spain, Ireland, France, Germany, Italy and Poland). There are two main types of equity release products the loan model and the sale model. According to (Dębicka and Marciniuk, 2014; Marciniuk, 2017; Marciniuk et al., 2020a), both products are available to individuals in Poland, i.e. a reverse mortgage (the loan model) and reverse annuity contract (the sale model). (Dębicka et al., 2020), (Dębicka and Marciniuk, 2014), (Marciniuk, 2017), (Marciniuk et al., 2020) and (Zmyślona and Marciniuk, 2020) consider also contracts for marriage. (Dębicka et al., 2020) and (Marciniuk, 2016) distinguished the dependence between the future lifetimes of spouses. In (Marciniuk, 2017) the reverse annuity is applied to derive two lemmas used to determine marriage benefits payable more than once a year inter alia in the case of Last Surviving Status and Joint Life Status. In the above papers, constant or depending on time, the interest rate is applied. (Marciniuk, 2021) consider an interest rate that varies from year to year. Different interest rate models are widely discussed in the literature. (Kellison, 2009) and (Boyle, 1976) used random variables as interest rates. Autoregressive processes are applied to actuarial science as technical interest rate models (Bellhouse and Panjer, 1981; Marciniuk, 2004; Panjer and Bellhouse, 1980). (Beekman and Fuelling, 1990, 1993; Dębicka, 2003; Garrido, 1988; Marciniuk, 2004; Parker, 1994a, 1994b) described the Wiener and Ornstein-Uhlenbeck process as an interest rate model. An extensive application of the short-term interest rate and description of the models can be found in the works of (Carriere, 1999, 2004; Jakubowski et al., 2003; Musiela and Rutkowski, 1988). (Dhaene, 2000) applies the CIR model to life insurance. (Carriere, 1999, 2004) considers stochastic interest rate models to determine actuarial values of life annuities and net premiums in life insurance. In actuarial science, the fuzzy set theory has been used to model problems connected with subjective judgment and situations when information available is imprecise and incomplete. We can meet articles on general actuarial issues using fuzzy sets in life and non-life insurance (Derrig and Ostaszewski, 2004; Lamaire, 1990; Ostaszewski, 1993; Shapiro, 2004). Life insurance issues such as calculation price life insurance policies, life insurance portfolios, life contingencies, life actuarial liabilities, life annuities have been covered in (Andres-Sanchez, Gonzalez-Vila Puchades and Gonzalez-Vila Puchades L., 2012; Andres-Sanchez and Gonzalez-Vila Puchades, 2017a; Andres-Sanchez and Gonzalez-Vila Puchades, 2017b) articles. (Derrig and Ostaszewski, 1997) dealt with issues related to property-liability insurance and (Shapiro, 2013) dealt with modelling future lifetime. The problems connected with risk models as ruin theory and claims aggregation were investigated in (Heilpern, 2018; Huang et al., 2009). The fuzzy interest rate was dealt in (Andres and Terceno, 2003; Betzuen et al., 1997; Lamaire, 1990).

2 METHODS

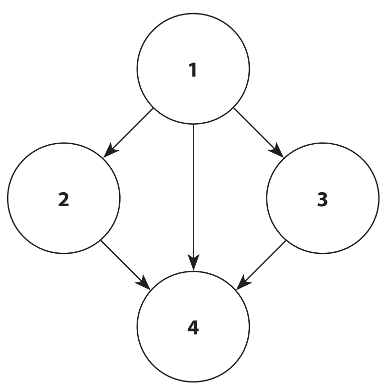
2.1 Matrix representation of benefits

2.1.1 Multiple state model

Multiple state modelling is a classical tool for designing and implementing insurance contracts. Among others it is used to calculate premiums for marriage insurances. We used the multistate model for marriage insurances to determine the benefits resulting from the equity release contract.

Generally the pair is called a multiple state model (MSM). S is the set space, where each state corresponds to an event which determines the cash flows (premiums and/or benefits). T denotes the set of direct transitions between states of S. For marriage reverse annuity contract, MSM has the following form:

Figure 1 Scheme of the multiple state model for marriage reverse annuity contract



State space:
1 - both spouses are alive
2 - husband is dead
3 - wife is dead
4 - both spouses are dead

Source: Adopted from (Denuit et al., 2001)

$$(S,T) = (\{1,2,3,4\},\{(1,2),(1,3),(1,4),(2,4),(3,4)\}), \tag{1}$$

and describes all possible contracted risk events up to the end of the contract. MSM given by Formula (1): is graphically presented in Figure 1, where states are marked with circles, and arrows indicate possible direct transitions between them.

In order to determine benefits it is necessary to formalize the description of probabilistic structure of the multiple state model, and all cash flows arising from the contract. Because the cash flows are realized at a certain time, it is necessary to choose an appropriate interest rate model. In this section we briefly discuss all three of these elements necessary for the valuation of benefits and finally we present the formulas for the annuity installment.

2.1.2 Probabilistic structure

The value of benefit is determine on the basis of spouses future lifetime, which depends on age at entry and the period of the contract. By x_w we denote the wife's age at entry and by x_m the husband's age at entry. We will assume that *the future life time of husband and wife are independent* (ASSUMPTION 1). In this paper, we consider a contract issued at time 0 and terminating according to the plan at a later time n, which is called the term of contract or the contract period. The length of the contract depends on its type. We will consider two types of contract, the last surviving status (LSS) when the annuity is paid as long as one spouse is alive and the joint-life status (JLS) when benefits are paid only while both spouses are alive. Thus the period of the contract n is defined as follows:

$n = \omega - min\{x_w, x_m\}$ for LSS,
$n = min\{\omega - x_w, \omega - x_m\}$ for JLS,

where ω is the maximum life expectancy which varies between 100 and 110 years depending on the population.

For a given contract, the function $Y(k)$ means that at time $k = \{0, 1, 2, ..., n\}$ (meaning the time that has elapsed from the beginning of the contract) the marriage is in one of four life situations described by the multiple state model shown in Figure 1. The life cases covered by the contract are random in nature. So it is natural to assume that $\{X(k); k = 0, 1, 2, ..., n\}$ is a discrete-time stochastic process taking values from a set space $S = \{1, 2, 3, 4\}$ and used to describe the evolution of the insured risk. Consistent with the actuarial literature, we assume that $\{X(k)\}$ is modelled by a nonhomogeneous Markov chain cf. (Hoem, 1969 and 1988; Norberg, 2002; Wolthuis, 1994). Because we focus on discrete-time model, where cash flows are made at the ends of time interval, the probabilistic structure of the model can be described in the matrix form:

$$\mathbf{D} = \begin{pmatrix} pr_1(0) & pr_2(0) & pr_3(0) & pr_4(0) \\ pr_1(1) & pr_2(1) & pr_3(1) & pr_4(1) \\ \vdots & \vdots & \vdots & \vdots \\ pr_1(n) & pr_2(n) & pr_3(n) & pr_4(n) \end{pmatrix}, \tag{2}$$

where $pr_i(k) = P(X(k) = i)$ and $i \in S$. Notice that each row of the matrix $\mathbf{D} \in \mathbb{R}^{(n+1)\times 4}$ is one-dimensional distribution for particular moment of the contract's period.

2.1.3 Cash flows

The individual's presence in a given state or movement (transition) from one state to another may have some financial impact. So, as a result of the agreement arise financial streams, consisting of cash flows of the agreement between the parties at the time. We consider two types of cash flow:

- a single payment (inflow from the company's mortgage fund point of view):

$$\pi_j(k) = \left\{ \begin{matrix} \alpha W & \text{for } j = 1 \text{ i } k = 0 \\ 0 & \text{besides} \end{matrix} \right\}, \tag{3}$$

where W is the value of property (benefits are calculated for a percentage α of the value of property, where $0 < \alpha \leq 0{,}5$ cf. (Dębicka and Marciniuk, 2014)),

- an annuity benefit i.e. a fixed annuity installment payable in advance which is determined depending on the type of contract. (outflow from the company's mortgage fund point of view). Depending on a status, annuity benefits $\ddot{b}$ are paid in more than one state (LSS):

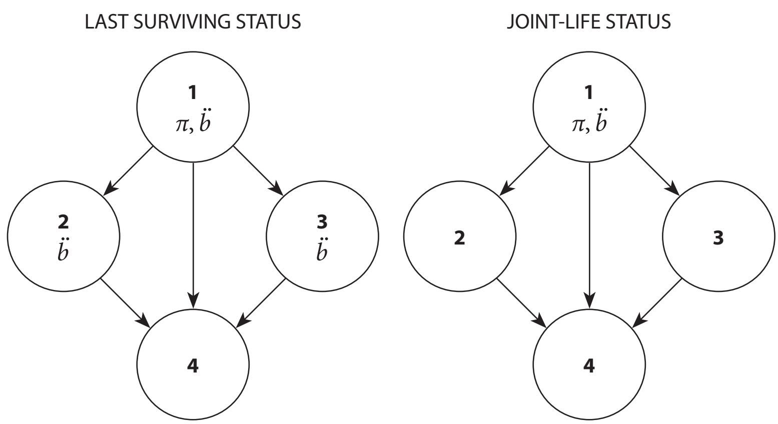
$$\ddot{b}_j(k) = \begin{cases} \ddot{b} & \text{for } j = 1 \text{ and } k = 0, 1, ..., n-1 \\ & \text{for } j = 2, 3 \text{ and } k = 1, 2, ..., n-1 \\ 0 & \text{besides} \end{cases}, \tag{4}$$

or only in one state (JLS):

$$\ddot{b}_j(k) = \begin{cases} \ddot{b} & \text{for } j = 1 \text{ and } k = 0, 1, ..., n-1 \\ 0 & \text{besides} \end{cases}. \tag{5}$$

Figure 2 illustrates the cash flows for the different statuses of the marriage reverse annuity contract.

Figure 2 MSM and cash flows for statuses of the marriage reverse annuity contract



Source: Based on (Dębicka et al., 2020)

Depending on a status, annuity benefits are paid in different. Therefore, we need to define the cash flow matrices for each status separately. Note that for the LSS contract, the annuity is paid when the process $\{X(k)\}$ is in states 1, 2 and 3, and then the cash flow matrix $\mathbf{C} \in \mathbb{R}^{(n+1)\times 4}$ has the following form:

$$\mathbf{C} = \begin{pmatrix} \alpha W - \ddot{b} & 0 & 0 & 0 \\ -\ddot{b} & -\ddot{b} & -\ddot{b} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\ddot{b} & -\ddot{b} & -\ddot{b} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha W & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\ddot{b} & 0 & 0 & 0 \\ -\ddot{b} & -\ddot{b} & -\ddot{b} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\ddot{b} & -\ddot{b} & -\ddot{b} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{C}_{in} + \mathbf{C}_{out}. \tag{6}$$

Whereas, for the JLS contract, the annuity is paid when the process {X(k)} is in state 1, and then the cash flow matrix $\mathbf{C} \in \mathbb{R}^{(n+1)\times 4}$ is as follows:

$$\mathbf{C} = \begin{pmatrix} \alpha W - \ddot{b} & 0 & 0 & 0 \\ -\ddot{b} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\ddot{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha W & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\ddot{b} & 0 & 0 & 0 \\ -\ddot{b} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\ddot{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{C}_{in} + \mathbf{C}_{out}. \tag{7}$$

Let us note that in Formulas (6) and (7) matrices $\mathbf{C}_{in}$ contain only cash flows which are inflow to the company's mortgage fund and $\mathbf{C}_{out}$ matrices are defined based on cash flows that are outflow from the company's mortgage fund.

2.1.4 Discount function

Interest rate may change with time, so it can be a function depending on time or can be modelled by stochastic process. Let $Y(t)$ denote the rate of interest in time interval [0, t]. Then $\{Y(t); t \geq 0\}$ is the interest rate accumulation process (stochastic process with stationary increments). From a technical point of view we assume *all moments of the random discounting function $e^{-Y(t)}$ are finite* (ASSUMPTION 2) cf. (Dębicka, 2013). Moreover, the future life time of spouses are independent on the interest rate and that means *the random variables $X(t)$ and $Y(t)$ are independent* (ASSUMPTION 3) cf. (Frees, 1990). For the discrete-time model, we define matrix $\mathbf{\Lambda} = [\lambda_{k_1k_2}]_{k_1,k_2=0}^{n} \in \mathbb{R}^{(n+1)\times(n+1)}$ based on the discount function, where:

$$\lambda_{k_1k_2} = E(e^{-(Y(k_1)-Y(k_2))}) = \begin{cases} E(v(k_2, k_1)) & \text{dla} \quad k_1 > k_2 \\ 1 & \text{dla} \quad k_1 = k_2 \\ E(r(k_1, k_2)) & \text{dla} \quad k_1 < k_2 \end{cases}. \tag{8}$$

For more on modelling the process $Y(t)$, and hence determining the elements of matrix $\mathbf{\Lambda}$, see Section 2.

2.1.5 Benefits

In Theorem 1 we present formulas for the annuity payable in advance for both statuses.

Theorem 1. Assume that the principle of equivalence and ASSUMPTIONS 1–3 are satisfied. For n-year marriage reverse annuity contract the cash flow matrix is determined for company's mortgage fund and αW is the capital for which the value of benefit is calculated. Let $\ddot{b}$ denote the annuity payable in advance if:

- both spouses are alive (JLS), then:

$$\ddot{b} = \frac{\mathbf{I}_1^T \mathbf{C}_{in} \mathbf{J}_1}{\mathbf{I}_1^T \mathbf{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n+1} \mathbf{I}_{n+1}^T) \mathbf{D} \mathbf{J}_1}, \tag{9}$$

- at least one of the spouses is alive (LSS), then:

$$\ddot{b} = \frac{\mathbf{I}_1^T \mathbf{C}_{in} \mathbf{J}_1}{\mathbf{I}_1^T \boldsymbol{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n+1} \mathbf{I}_{n+1}^T - \mathbf{I}_1 \mathbf{I}_1^T) \mathbf{D}(\mathbf{S} - \mathbf{J}_4) + 1}, \tag{10}$$

where the vectors in (9) and (10) are defined as follows: $\mathbf{S} = (1, 1, 1, 1)^T$, $\mathbf{J}_1 = (1, 0, 0, 0)^T$, $\mathbf{J}_4 = (0, 0, 0, 1)^T$, $\mathbf{I}_1 = (1, 0, ..., 0)^T \in \mathbb{R}^{(n+1)}$ and $\mathbf{I}_{n+1} = (0, ..., 0, 1)^T \in \mathbb{R}^{(n+1)}$.

We drop the proof of the theorem on the grounds that it is analogous to the proofs of the theorems presented in (Dębicka, 2013; Dębicka et al., 2020).

Note that, since the matrix $\mathbf{C}_{in}$ (which depends only on cash flow arising from the contract) and matrix $\mathbf{D}$ (which depends on the probabilistic structure of the MSM) are constant for a given contract, the amount of the annuity instalment depends on the choice of the interest rate model (forms of the matrix $\boldsymbol{\Lambda}$).

It is easy to observe that the numerators in Formulas (9) and (10) are equal to the corresponding percentage of the value of property, that is $\mathbf{I}_1{}^T\mathbf{C}_{in}\mathbf{J}_1 = \alpha W$. Moreover, the denominators correspond to the unit annuities payable in advance (temporary life annuity due) $\ddot{a}_{\overline{x_w x_m:n|}}$:

- for JLS (the unit annuity instalment is paid only if both spouses are alive – in state 1) we have $\ddot{a}_{\overline{x_w x_m:n|}} = \mathbf{I}_1^T \boldsymbol{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n+1} \mathbf{I}_{n+1}^T)\mathbf{D}\mathbf{J}_1$,

- for LSS (the annuity instalment is paid only if at least one spouse is alive – in states 1, 2, 3) we have $\ddot{a}_{\overline{x_w x_m:n|}} = \mathbf{I}_1^T \boldsymbol{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n+1} \mathbf{I}_{n+1}^T - \mathbf{I}_1 \mathbf{I}_1^T) \mathbf{D}(\mathbf{S} - \mathbf{J}_4) + 1$.

The formulas in Theorem 1 can also be used when annuities are paid more frequently than annually. Let m be the frequency of annuity payments, e.g. $m = 2$ (half-yearly annuity), $m = 4$ (quarterly annuity), $m = 12$ (monthly annuity). The annual annuity installment payable m times a year is thus given by the formula:

$$\ddot{b}^{(m)}_{\overline{x_w x_m:n|}} = \frac{\alpha W}{\ddot{a}_{\overline{x_w x_m:n|}}} = m \cdot \frac{\alpha W}{\ddot{a}_{\overline{x_w x_m:n \cdot m|}}}, \tag{11}$$

where $\ddot{a}_{\overline{x_w x_m:n|}}$ is the unit annual annuity due paid m times a year in the amount of $\frac{1}{m}$:

$$\ddot{a}^{(m)}_{\overline{x_w x_m:n|}} = \begin{cases} \frac{1}{m} \sum_{k=1}^{nm-1} \mathrm{E}(v(0,\frac{k}{m}))pr_1(k/m) & \text{for JLS} \\ \frac{1}{m}\ 1 + \sum_{k=1}^{nm-1} \mathrm{E}(v(0,\frac{k}{m}))(pr_1(k/m) + pr_2(k/m) + pr_3(k/m))) & \text{for LSS}, \end{cases} \tag{12}$$

where $pr_i(k/m) = pr(X(k/m) = i)$ for $k = 0, 1, 2, ..., nm - 1$ and $i = 1, 2, 3$. Note that from (12) we have $\ddot{a}_{\overline{x_w x_m:n|}} = \frac{1}{m}\ddot{a}_{\overline{x_w x_m:n \cdot m|}}$ where $\ddot{a}_{\overline{x_w x_m:n \cdot m|}}$ is the unit annuity payable in advance for the period and can be determined in a matrix manner as in Theorem 1:

$$\ddot{a}_{\overline{x_w x_m:n \cdot m|}} = \begin{cases} \mathbf{I}_1^T \boldsymbol{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n \cdot m+1} \mathbf{I}_{n \cdot m+1}^T)\mathbf{D}\mathbf{J}_1, & \text{for JLS} \\ 1 + \mathbf{I}_1^T \boldsymbol{\Lambda}^T (\mathbf{I} - \mathbf{I}_{n \cdot m+1} \mathbf{I}_{n \cdot m+1}^T - \mathbf{I}_1 \mathbf{I}_1^T) \mathbf{D}(\mathbf{S} - \mathbf{J}_4) & \text{for LSS}. \end{cases} \tag{13}$$

Note that, increasing the frequency of annuity payments will increase the dimension of the matrices that depend on the insurance period n which are used in (13). The dimensions of these matrices will grow up into $n \cdot m+1$. In particular matrix $\mathbf{D} \in \mathbb{R}^{(n \cdot m+1)\times 4}$ and its elements will be determined under the assumption that the probability of death within a year is uniform. Similarly, the matrix $\mathbf{\Lambda} \in \mathbb{R}^{(n \cdot m+1)\times(n \cdot m+1)}$ and its elements will be determined for a correspondingly shorter interest rate.

Additionally, let us note that in (11) the factor $\frac{\alpha W}{\ddot{a}_{\overline{x_w x_m : n \cdot m|}}}$ is the amount of the monthly annuity due paid $m \cdot n$ times.

2.2 Interest rate modelling

There are three possibilities for modelling the interest rate: the actuarial, financial and fuzzy. The first two ways are described in detail in (Marciniuk, 2009). These ways assume that the interest rate is a stochastic process and require knowledge about the technical and short-term rates. The third possibility considers the constant interest rate modelled by a fuzzy number. The necessary concepts will be introduced below and the interest rate models used in the empirical section.

2.2.1 Actuarial and financial modelling of interest rate

At the beginning, it is necessary to explain the relationship between actuarial and financial approaches to the interest rate modelling.

The actuarial way requires an understanding of a technical discounting function. Let us assume that capital K_t is invested at the moment t. Its value after T ($0 \leq t \leq T$) years is K_T. The discounting function from time T on time t is defined as follows (Marciniuk et al., 2020):

$$v_{t,T} = \frac{K_T}{K_t}, \ 0 \leq t \leq T. \tag{14}$$

The denotation in the financial literature of the discounting function $v_{t,T}$ is equivalent to that adopted in Section 2.1, i.e. $v(t,T)$ (c.f. Formula (8)).

The discounting function has the following relationship with the force of interest rate function $\delta_{t,T}$ as follows (Bellhouse and Panjer, 1981):

$$v_{t,T} = exp(-\delta_{t,T}), \quad 0 \leq t \leq T, \tag{15}$$

therefore in the actuarial approach the discount function or the force of interest rate can be modelled.

To determine financial models of an interest rate it is necessary to introduce the concept of pricing a zero-coupon bond (Marciniuk et al., 2020). A zero-coupon bond is a stock, which is sold at a discount. The customer's profit is the difference between its nominal and selling price (Musiela and Rutkowski, 1988). By convention, the nominal price is one financial unit. It means that the zero-coupon's holder will receive one unit of cash at moment T. The price of a zero-coupon bond of maturity T at any instant t ($0 \leq t \leq T$) is denoted by $P_{t,T}$. It is obvious that $P_{t,T} = v_{t,T}$, because:

$$v_{t,T} = \frac{K_t}{K_T} = \frac{P_{t,T}}{P_{T,T}} = \frac{P_{t,T}}{1} = P_{t,T}. \tag{16}$$

Therefore, the discounting function could be described as the price of a zero-coupon bond (Marciniuk, 2009) and could be modelled by the use of $P_{t,T}$.

In the actuarial approach, the models of the force interest rate function $\delta_{t,T}$ from time t to time T, $0 \leq t \leq T$, are applied. The force of interest δ_t at the moment $t \geq 0$ is described using a stochastic process with discrete or continuous time. Hence:

$$\delta_{t,T} = \begin{cases} \sum_{t}^{T} \delta_t, & when \quad s = 0, 1, 2, ... \\ \int_{t}^{T} \delta_t ds, & when \quad s \geq 0 . \end{cases} \tag{17}$$

The most popular stochastic model of the force of interest is a process with a discrete-time called an autoregressive process of order one (AR(1)). This process is defined by the following recursive relation (Brockwell and Davis, 1996):

$$\delta_t = \mu + \phi(\delta_{t-1} - \mu) + \varepsilon_t, \quad t = 1, 2, ... , \tag{18}$$

where $\delta_0 \in \mathbb{R}, \mu \in \mathbb{R}, |\phi| < 1$. Moreover $\varepsilon_t \sim N(0,\sigma^2)$ and variables δ_t and ε_s are independent for $s < t$. This process is stationary and has a normal distribution with mean μ and variance $\frac{\sigma^2}{1-\phi^2}$.

To calculate the benefit of reverse marriage annuity, it is necessary to know the value $E(v_{0,t}^k)$. It can be concluded from the fact that the process $\{\delta_t\}_{t\geq 0}$ has a normal boundary distribution, that the interest rate function $\delta_{t,T}$ also has a normal distribution. Hence and from Formula (16), it follows that (Panjer and Bellhouse, 1980):

$$E(v_{0,t}^k) = M_{\delta_{0,t}}(-k) = exp(-k\mu t + 0,5k^2\, V(\delta_{0,t})) , \tag{19}$$

where:

$$V(\delta_{0,t}) = \frac{t\sigma^2}{1-\phi} + 2\,\frac{\sigma^2}{1-\phi^2}\,\frac{\phi}{1-\phi}\,(t - 1 - \phi\,\frac{1-\phi^{t-1}}{1-\phi}) . \tag{20}$$

$M_X(k)$ means the function generating the moments of the variable X in point k.

The autoregressive process of order one is a discrete version of the continuous Ornstein-Uhlenbeck process.

The process, which is applied as a continuous force of interest $\{\delta_t\}_{t\geq 0}$, is the following stochastic Wiener process (Dhaene, 2000):

$$d\delta_t = \sigma dB_t, \qquad t \geq 0, \tag{21}$$

where $\delta_0 \in \mathbb{R}$, $\sigma \geq 0$ and $\{B_t\}_{t\geq 0}$ mean the standard Brownian motion.

The solution of the above stochastic differential equation is a process with the following form:

$$\delta_t = \delta_0 + \sigma B_t, \quad t \geq 0. \tag{22}$$

Hence it is easy to prove that (Musiela and Rutkowski, 1988):

$$E(\delta_t) = \delta_0, \quad V(\delta_t) = \sigma^2 t, \quad C(\delta_s,\delta_t) = \sigma^2 C(B_s,B_t) = \sigma^2 min(s, t).$$

Because process $\{\delta_t\}_{t\geq 0}$ is also a Gaussian process, the following equity is accurate:

$$E(v_{0,t}^k) = M_{\delta_{0,t}}(-k)\ . \tag{23}$$

In this case $M_{\delta_{0,t}}(k)$ is given as follows (Marciniuk, 2004):

$$M_{\delta_{0,t}}(k) = exp(kt\delta_0 + k_2\frac{\sigma^2 t^3}{6})\ . \tag{24}$$

The price of a zero-coupon bond can be determined using the short-term interest rate process $\{r_t\}_{t\geq 0}$ (Carriere, 2004; Musiela and Rutkowski, 1988). If r_t is a stochastic process adaptive with the filtering F_t, $\int_0^T |r_s| ds < \infty$ and a martingale measure **Q** equivalent to the measure **P** exists on the probabilistic space $(\Omega, F, \mathbf{P})$, then:

$$P_{t,T} = E^Q(exp(-\int_t^T r_s ds)|F_t)\ . \tag{25}$$

Consider the case when the short-term rate is determined by the following Ornstein-Uhlenbeck process (Vasicek, 1999):

$$dr_t = -\alpha(r_t - \mu)dt + \sigma dB_t\ , \tag{26}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $\alpha > 0$.

Process $\{B_t\}_{t\geq 0}$ is a standard Brownian motion under the **Q** measure. From the Girsanov theorem, it is known that $B_t = B_t^* + B_t$, where $\{B_t^*\}_{t\geq 0}$ is the standard Brownian motion under the **P** measure, where β ($\beta > 0$) means the price of the risk (Carriere, 2004; Jakubowski et al., 2003; Vasicek, 1999). This model of the short-term rate is known as a Vasicek model.

The following formula gives the price of a zero-coupon bond at the moment 0 with the maturity t ($t \geq 0$):

$$P_{0,t} = exp(-r_0 \cdot n_{0,t} + m_{0,t}), \tag{27}$$

where:

$$n_{0,t} = \frac{1}{\alpha}\ n_{0,t}(1 - e^{-\alpha t}), \tag{28}$$

$$m_{0,t} = -\ \mu t + \mu \cdot n_{0,t} + 0.5\sigma^2(\frac{t}{\alpha^2} + \frac{2}{\alpha^3}(1 - e^{-\alpha t}) - 0.5\frac{1}{\alpha^3}(1 - e^{-2\alpha t})). \tag{29}$$

Consider the Cox-Ingersoll-Ross (CIR) model, when the short-rate is determined by the use of the stochastic differential equation (Dhaene, 2000; Jakubowski et al., 2003; Musiela and Rutkowski, 1988):

$$dr_t = \alpha(\mu - \alpha r_t)dt + \sigma\sqrt{r_t dB_t}\ , \tag{30}$$

where $\mu \in \mathbb{R}$, $0 < \sigma < \alpha$, $B_t = B_t^* + \int_0^t \sqrt{r_u}\, du$.

Formula (27) gives the price of a zero-coupon bond, where (Denuit et al., 2001; Jakubowski et al., 2003; Marciniuk, 2009):

$$n_{0,t} = \frac{2(exp(2\gamma t) - 1)}{2\gamma - \alpha + (2\gamma + \alpha)\ exp(2\gamma t)}\ , \tag{31}$$

$$m_{0,t} = \frac{2\mu}{\sigma^2}\ ln(\frac{4\gamma\ exp(0.5(2\gamma + \alpha)t)}{2\gamma - \alpha + (2\gamma + \alpha)\ exp(2\gamma t)})\ , \tag{32}$$

and $\gamma = 0.5\ \sqrt{\alpha^2 + 2\sigma^2}$.

2.2.2 Fuzzy interest rate

First, we recall some notions connected with fuzzy sets. The fuzzy subset **A** of the space X is described by its membership function μ**A**: $X \rightarrow [0, 1]$ (Dubois and Prade, 1980; Zadeh, 1965). The value $\mu_{\mathbf{A}}(x)$ represents the degree of membership of the element x to the fuzzy set **A**. If $\mu_{\mathbf{A}}(x) = 0$ then we have non-membership of x and for $\mu_{\mathbf{A}}(x) = 1$ we obtain absolute membership. The crisp sets $A\alpha = \{x \in X \mid \mu_{\mathbf{A}}(x) \geq \alpha\}$ for any $0 < \alpha \leq 1$ are called the α-cuts and the support $A0$ of the fuzzy set **A** is the closure of set $\{x \in X \mid \mu_{\mathbf{A}}(x) > 0\}$. The set A_1 is called the core of **A**. The α-cuts $A\alpha$ unambiguously define the fuzzy set **A**. We will denote the fuzzy sets by the bold letters, e.g. **A**, and the crisp sets by the italic, non-bold letters, e.g. $A\alpha$.

In our paper, we will use the fuzzy subsets **A** of real line $\mathbb{R}$, i.e. $X = \mathbb{R}$. In addition, we assume that $A_0 = [a, d]$, $A_1 = [b, c]$ and the membership function $\mu_{\mathbf{A}}$ is continuous. Moreover, this function is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$. Every α-cuts of such fuzzy set is the compact interval, i.e. $A_\alpha = [A_\alpha^L, A_\alpha^U]$ and this fuzzy set is called the fuzzy number (Dubois and Prade, 1980). We obtain the trapezoidal fuzzy number when the membership function on the intervals $[a, b]$ and $[c, d]$ are linear. We denote it as $\mathbf{A} = (a, b, c, d)$. If $a = b$ we have the triangular fuzzy number $\mathbf{A} = (a, b, d)$. The actual number a can be treated as the degenerate triangular fuzzy number (a, a, a).

In our paper, we will use the arithmetic operation on fuzzy numbers. Moreover, the fuzzy numbers used in this paper are positive, i.e. $a > 0$. These arithmetic operations are based on the α-cuts and they are defined in the following way:

$$(A + B)_\alpha^{\ L} = A_\alpha^{\ L} + B_\alpha^{\ L},\ (A + B)_\alpha^{\ U} = A_\alpha^{\ U} + B_\alpha^{\ U}\ , \tag{33}$$

$$(A \cdot B)_\alpha^{\ L} = A_\alpha^{\ L} \cdot B_\alpha^{\ L},\ (A \cdot B)_\alpha^{\ U} = A_\alpha^{\ U} \cdot B_\alpha^{\ U}\ , \tag{34}$$

$$(A\ /\ B)_\alpha^{\ L} = A_\alpha^{\ L}\ /\ B_\alpha^{\ U},\ (A\ /\ B)_\alpha^{\ U} = A_\alpha^{\ U}\ /\ B_\alpha^{\ L}\ , \tag{35}$$

$$(\lambda A)_\alpha^{\ L} = \begin{cases} \lambda A_\alpha^{\ L} \text{ for } \lambda \geq 0 \\ \lambda A_\alpha^{\ U} \text{ for } \lambda < 0 \end{cases},\ (\lambda A)_\alpha^{\ U} = \begin{cases} \lambda A_\alpha^{\ U} \text{ for } \lambda \geq 0 \\ \lambda A_\alpha^{\ L} \text{ for } \lambda < 0\ , \end{cases} \tag{36}$$

$$(A^\alpha)_\alpha^{\ L} = (A_\alpha^{\ L})^\alpha \quad (A^\alpha)_\alpha^{\ U} = (A_\alpha^{\ U})^\alpha\ . \tag{37}$$

The sum of the fuzzy triangular numbers **A** + **B** is the triangular fuzzy number, too. This property also holds for the product λ**A**. The results of other arithmetic operations are the fuzzy number, but not fuzzy triangular numbers, the membership functions are not linear. We can define the difference of the fuzzy numbers **A** – **B** using α-cuts or the formula:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}. \tag{38}$$

The area below the graph of membership function of the fuzzy number can be treated as the degree of imprecision and it is denoted as Imp(**A**). For the triangular fuzzy number $\mathbf{A} = (a, b, d)$ we obtain:

$$Im\,\mathrm{p(A)} = \frac{d-a}{2}\,. \tag{39}$$

The mean value of the fuzzy number **A** is defined as follows:

$$M(\mathbf{A}) = \frac{1}{2}\int_0^1 (A_\alpha^{\ L} + B_\alpha^{\ U})\, d\alpha\,. \tag{40}$$

We have:

$$M(\mathbf{A}) = \frac{a + 2b + d}{4} \tag{41}$$

for the triangular fuzzy number **A**.

The triangular fuzzy number $\mathbf{A} = (a, b, d)$ has a linear membership function. So, the parameters a, b and d univocally define it.

We can define the order between triangular fuzzy numbers in the following, natural way. Let $\mathbf{A_1} = (a_1, b_1, d_1)$ and $\mathbf{A_2} = (a_2, b_2, d_2)$ than:

$$\mathbf{A_1} \le \mathbf{B_2} \Leftrightarrow (a_1 \le a_2, b_1 \le b_2, d_1 \le d_2)\,. \tag{42}$$

Now we investigate the case when the interest rate has imprecision form. For instance, we obtain the imprecision information, that it is "about 0.055". We can model such interest rate as the triangular fuzzy number $\mathbf{i} = (a, b, d)$. The parameters of this fuzzy number are determined based on additional information, experts' assessments, and own intuition. Let us assume that fuzzy interest rate takes the form:

$$\mathbf{i} = (0.05, 0.055, 0.065)\,. \tag{43}$$

The mean value of this fuzzy number is equal $M(\mathbf{i}) = 0.05625$ and imprecision Imp(**i**) = 0.0075.

The graph of the membership function of triangular fuzzy number **i**, the sample α-cut I0.4 and the mean value $M(\mathbf{i})$ are included in Figure 7a.

The fuzzy number 1 + **i**, where 1 is treated as a degenerate triangular fuzzy number (1, 1, 1), is triangular too. We have 1 + **i** = (1.05, 1.055; 1.065). The fuzzy discounting factor $\mathbf{v} = 1/(1 + \mathbf{i})$ is not a triangular fuzzy number. Every α-cuts of it take a form:

$$v_\alpha = [(1.065 - 0.01\alpha)^{-1},(1.05+0.005\alpha)^{-1}]\,. \tag{44}$$

So, it is the fuzzy number with the following membership function:

$$\mu_{\text{v}}(x) = \begin{cases} (1.065x - 1) / 0.01x & \text{for } 1/1.065 \le x < 1/1.055 \\ (1 - 1.05x) / 0.005x & \text{for } 1/1.055 \le x \le 1/1.05 \\ 0 & \text{otherwise} \end{cases}. \tag{45}$$

The graph of such membership function is presented in Figure 7b. We see that it is almost linear, so we can treat it as a linear fuzzy number (0.939, 0.948, 0.952).

We can approximate the fuzzy discounting factor **v** as the triangular fuzzy number (v_a, v_b, v_d). The α-cut of fuzzy power $\mathbf{v}^c$, where $c > 0$, takes the form:

$$(v^c)_\alpha = [(v_a + (v_b - v_a)\alpha)^c, (v_d - (v_d - v_b)\alpha)^c]. \tag{46}$$

This fuzzy number is almost linear, too.

If we consider the joint-life status (JLS), then based on (12), we can treat the fuzzy joint-life annuity $\ddot{\mathbf{a}}^{(m)}_{\overline{x_w x_m : n|}}$ as the triangular fuzzy number.

$$(\frac{1}{m}\sum_{k=0}^{nm-1} v_a^{k/m} pr_1(\frac{k}{m}), (\frac{1}{m})\sum_{k=0}^{nm-1} v_b^{k/m} pr_1(\frac{k}{m}), (\frac{1}{m})\sum_{k=0}^{nm-1} v_d^{k/m} pr_1(\frac{k}{m}). \tag{47}$$

Considering (47) and taking into account (35), we derive the fuzzy annuity benefit paid m times a year:

$$\ddot{\mathbf{b}}^{(m)} = (\ddot{b}_a^{(m)}, \ddot{b}_b^{(m)}, \ddot{b}_d^{(m)}) = (\frac{\alpha W}{(\ddot{a}^{(m)}_{\overline{x_w x_m : n|}})_d}, \frac{\alpha W}{(\ddot{a}^{(m)}_{\overline{x_w x_m : n|}})_b}, \frac{\alpha W}{(\ddot{a}^{(m)}_{\overline{x_w x_m : n|}})_a}). \tag{48}$$

3 RESULTS

The short-term rate is not directly observed on the financial market, therefore to calculate the benefit of marriage reverse annuity contract, we assume that we know the simulated data of the short-term rate. The data was generated from the following distribution (James and Webber, 2000; Marciniuk, 2009):

$$r_{t_{i+1}} \mid r_{t_i} \sim N(\mu + (r_{t_i} - \mu)\, exp(-\alpha(t_{i+1} - t_i)), \sigma\sqrt{\frac{1 - exp(-2\alpha(t_{i+1} - t_i))}{2\alpha}}, \tag{49}$$

for $\alpha = 8$, $\mu = 0.055$, $\sigma = 0.04$, $r_0 = 0.05$.

We assume that the weekly data was observed throughout 20 years. The parameters of the interest rate models were estimated based on these data using the maximum likelihood method in the case of the Wiener process, AR(1) process and the Vasicek model. The general method of moments was applied in the case of the CIR model. In this aim, we use the packet Solver in Microsoft Excel. The results of the estimation are as follows (Marciniuk, 2009):

- AR(1) process:
  $d\delta_t = 0.05524 + 0.84598(\delta_{t-1} - 0.05524) + \varepsilon_t$,
  $\varepsilon_t \sim N(0, 0.009375)$, $\delta_0 = 0.04845$,
- Wiener process:

$d\delta_t = 0.0052 dB_t$, $\delta_0 = 0.04845$,

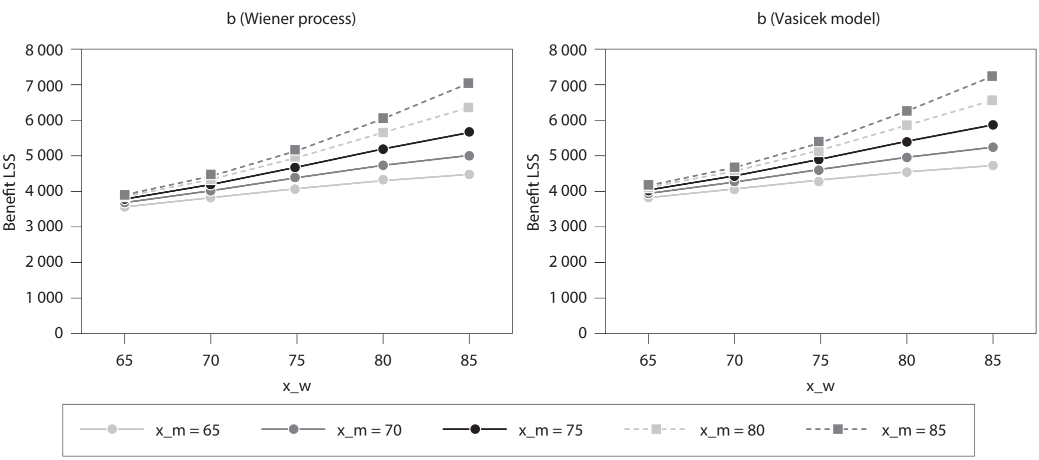
– Vasicek model:
$dr_t = -8.67(r_t - 0.055)\, dt + 0.04 dB_t$,

– CIR model:
$dr_t = (0.06218 - 1.1254 r_t)\, dt + 0.32\sqrt{r_t}\, dB_t$.

Moreover, we assume that the property's actual value W is 100 000 euros and a reduction factor α of 50%. We use Polish Life Tables from 2012. We take into account the uniform distribution within a year. The spouses' future lifetimes are independent random variables. The constant technical interest rate is equal to the long-term interest rate in the Vasicek model and is almost 5.5%. Hence the fuzzy interest rate is also approximated at about 5.5%.

At the beginning, we discuss actuarial and financial models. We distinguish between two statuses: a joint-life status (JLS), when the benefit is paid only until the death of the first spouse, and a last surviving status (LSS) contract by which the benefit is paid until the death of the other spouse (Dębicka et al., 2020). The benefits are calculated from Formulas (9) and (10) and their modification, described at the end of Section 2.1. The graph below shows the benefit amounts of the marital reversionary annuity for the Vasicek model and the Wiener process.

Figure 3 The LSS benefits in the case of the Wiener process and the Vasicek model

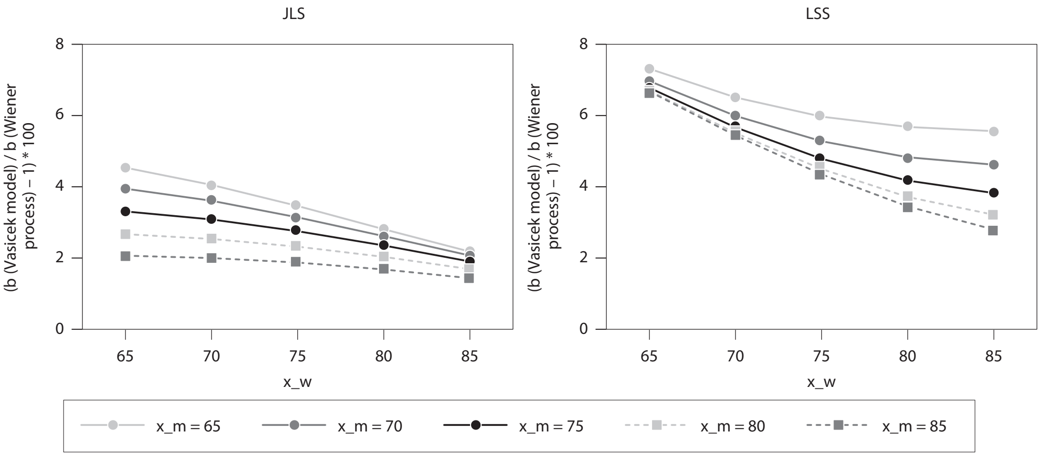


Source: Own elaboration

The Wiener process obtains the lowest benefits and the highest for the Vasicek model. Therefore in the following graphs, we can see the relative percentage differences between these benefits for the man at different ages x_m depending on woman's age x_w, calculated as follows:

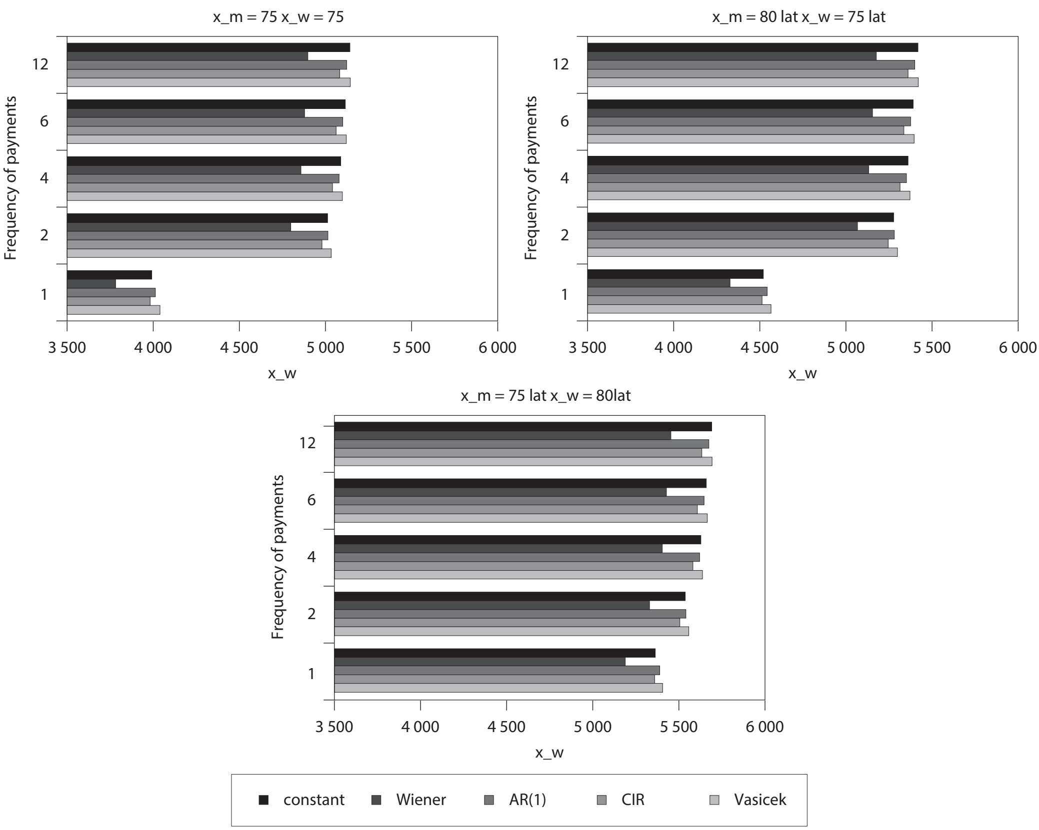
$$relativedifferences = \left(\frac{b(Vasicek\text{model})}{b(Wiener\text{process})} - 1\right) \cdot 100\% \,. \tag{50}$$

Figure 4 The relative percentage differences between benefits in the case of the Wiener process and the Vasicek model



Source: Own elaboration

Figure 5 Annual annuity benefits for all models

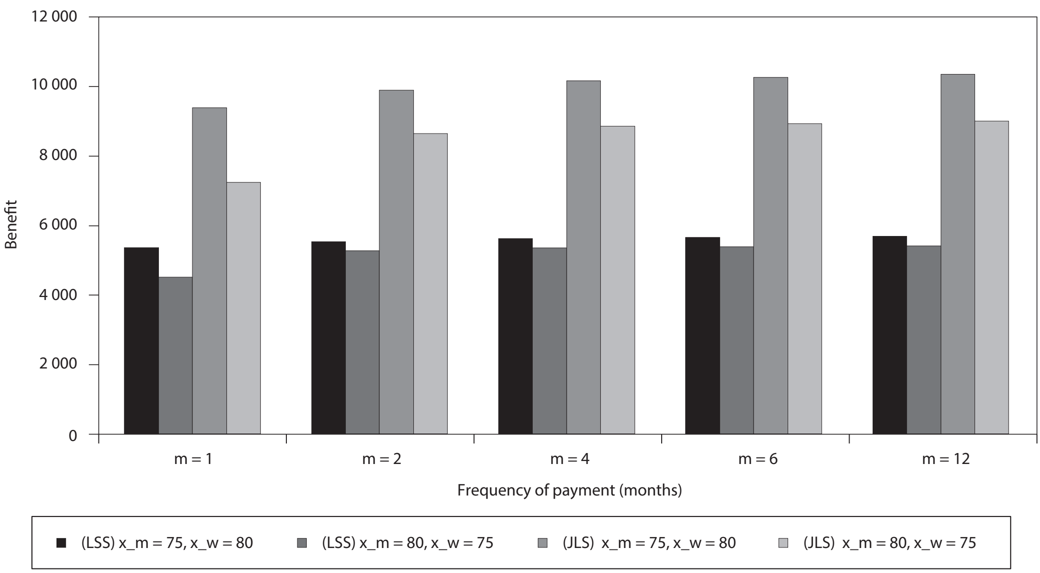


Source: Own elaboration

Figure 5 shows the differences in annuities for spouses of different ages in the case of LSS, when benefits are paid annually and more than once a year (m = 1, 2, 4, 6, 12).

For the joint-live status, the situation is similar, but the differences in benefit amounts are lower. It can be seen in Figure 6 for a fixed technical interest rate. Fogure 6 shows benefits for spouses of different ages for both statuses.

Figure 6 Annual annuity benefits for constant rate model



Source: Own elaboration

The differences in benefits between more frequent and annual payments are shown in Table 1.

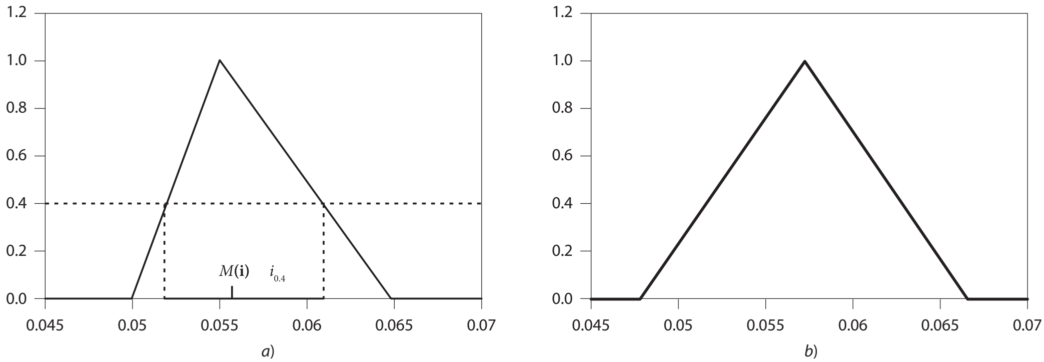
Table 1 The differences in benefits between more frequent and annual payments

	m **= 2**	***m*** **= 4**	***m*** **= 6**	***m*** **= 12**
(LSS) *x_m* = 75, *x_w* = 80	3.25%	4.96%	5.54%	6.13%
(LSS) *x_m* = 80, *x_w* = 75	19.49%	22.38%	23.37%	24.37%
(JLS) *x_m* = 75, *x_w* = 80	5.42%	8.32%	9.30%	10.30%
(JLS) *x_m* = 80, *x_w* = 75	14.43%	17.57%	18.64%	19.72%

Source: Own elaboration

For LSS the differences range from 3.25% (m = 2) to 6.13% and for JLS (m = 12) from 5.42% (m = 2) to 10.3% (m = 12) when x_m = 75 and x_w = 80. When x_w = 80 and x_m = 75 the differences are significantly higher than for younger male and older female, but for JLS they are lower than for LSS (JLS: from 14.43% (m = 2) to 19.72% (m = 12); LSS: from 19.49% (m = 2) to 24.37% (m = 12)). The differences in benefits paid six or twelve times a year compared to benefits paid respectively four or six times a year are 0.56%

Figure 7 The membership functions of fuzzy interest rate **i** (*a*) and fuzzy discounting factor **v** (*b*)



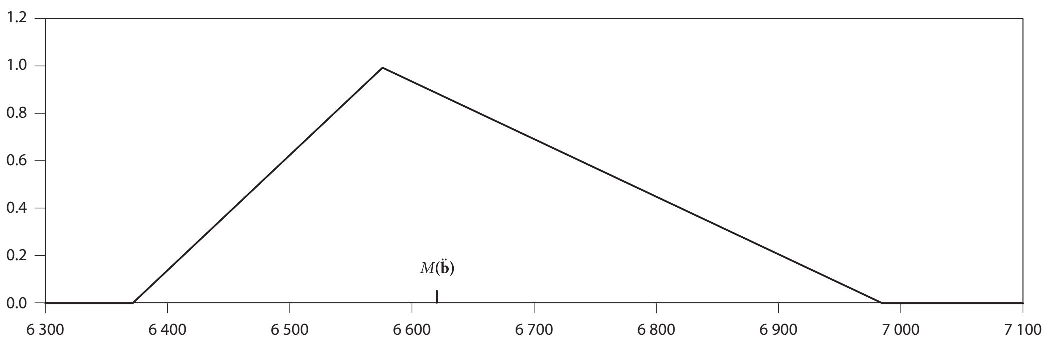
Source: Own construction

(LSS) and 0.91% (JLS) for younger men. The difference is slightly lower for younger women (0.54% and 0.81% respectively).

In Section 2.2.2 we computed the fuzzy interest rate **I** and the fuzzy discounting factor **v**. The graphs of such fuzzy numbers are presented in Figure 7.

Now, we derive the fuzzy annuity benefit $\ddot{\mathbf{b}}^{(m)}$. We assume that, man and woman are 65 years old, i.e. $x_m = x_w = 65$ and the joint-life annuity is paid monthly, i.e. $m = 12$ (compare Section 2.2). The fuzzy annual annuity benefit is the almost triangular fuzzy number (6 373.66, 6 576.82, 6 986.77) with mean value $M(\ddot{\mathbf{b}}^{(m)}) = 6\,628.52$ and imprecision $\mathrm{Imp}(\ddot{\mathbf{b}}^{(m)}) = 306.55$. The graph of the membership function of this fuzzy number and its mean value are included in Figure 8.

Figure 8 The membership function of fuzzy annual annuity benefit $\ddot{b}^{(m)}$



Source: Own construction

Table 2 contains the parameters of the fuzzy annual annuity benefits, the mean values and imprecisions of such fuzzy numbers, when the spouses are the same age and when the husband is 75 years old and the wife is of different ages.

Table 2 The parameters of the fuzzy annual annuity benefits for JLS

x_m	*x_w*	*a*	*b*	*c*	Mean	Imp
The same age of spouses						
65	65	6 373.66	6 576.82	6 986.77	6 628.52	306.55
70	70	7 653.35	7 856.37	8 264.86	7 907.73	305.76
75	75	9 670.28	9 876.48	10 290.06	9 928.32	309.89
80	80	12 831.30	13 044.39	13 470.50	13 097.64	319.60
85	85	17 553.49	17 774.78	18 216.27	17 829.83	331.39
The different age of spouses						
75	65	8 352.05	8 558.78	8 974.11	8 610.93	311.03
75	70	8 805.96	9 011.84	9 425.23	9 063.72	309.64
75	75	9 670.28	9 876.48	10 290.06	9 928.32	309.89
75	80	11 175.06	11 384.13	11 802.79	11 436.53	313.87
75	85	13 533.87	13 747.27	14 173.91	13 800.58	320.02

Source: Own elaboration

4 DISCUSSION

In the first part of Section 3, we discussed benefits under assumptions that actuarial and financial models model interest rates. Regardless of the status of the contract, it turned out that the benefit amounts of the marital reversionary annuity for the Wiener process obtain the lowest benefits and the highest for the Vasicek model. Figure 3 and Figure 4 illustrate the fact that the benefits increase and the differences decrease with the rise of spouses' age. However, the payment structure in the two statuses is various. The differences between benefits are higher for the younger spouses and lower for the older spouses in the last surviving status and vice versa in the joint-life status.

Apart from the interest rate model, the frequency of benefit payments significantly impacts their amount. It turns out that there are differences in annuities for spouses of different age in the case of last surviving status when benefits are paid annually and more than once a year. It can be observed (com. Figure 5) that the benefits are lowest in the case of the Wiener process. In other cases, they are of a similar amount. The more often the benefit is paid, the higher the benefit is, but the differences are insignificant for the higher than two months frequency. The most significant differences are between an annual benefit and a benefit paid every two months. For the joint-live status, the situation is similar, but the differences in benefit amounts are lower (com. Figure 6 and Table 1).

The age difference between the spouses also affects the amount of benefits. For both statuses, the differences are significantly higher for an older husband and younger wife than for a younger male and older female, but for JLS they are lower than for LSS.

In the second part of Section 3, we derived the fuzzy annuity benefit. Firstly, we considered the situation that the spouses are the same age. It turned out that as the age of spouses increases, the value of the fuzzy annual annuity benefit increases, too. For older spouses, the increase is more significant (com. Figure 8). We can also observe a slight increase in imprecision. Secondly, we analysed the fuzzy annual annuity benefits when the husband is 75 years old, and the wife is of different age (com. Table 2). We obtained similar precision as in the case that spouses are equal in age.

To summarize, the benefit amounts are highest for the Vasicek model, slightly lower benefits are obtained for the AR(1) process. The autoregressive process of order one is the discrete equivalent of the Ornstein-Uhlenbeck process (Vasicek model). Therefore, similar results are obtained for both models.

The lowest benefit is obtained for the Wiener process. As the frequency of benefits increases, the annual benefit increases. The benefit obtained at a constant interest rate is similar to the Vasicek model. The number of payments per year affects the annual benefit obtained, with withdrawals more frequent than 12 times per year no longer significantly improving the benefit. As the age of spouses increases, the value of the fuzzy annual annuity benefit increases, too. For older spouses, the increase is more significant. We can also observe a slight increase in imprecision. The fuzzy rate has not yet been applied to the determination of reverse annuity benefits.

CONCLUSION

Equity release contracts have long been widely discussed. These contracts are addressed to older people. Europe's ageing population, and therefore a rising dependency ratio of retirees on the working population, strongly suggests that a pensions funding gap will be a key social issue in the future. Equity release contracts that can fill this gap. Many older people own property and in return for a monthly benefit, they could access using equity release products.

The article sets out the benefits of a marriage reverse annuity contract, which exists in Poland only in individual form. Usually, the spouses own their property. That is why marriage contracts are a natural research issue. Various interest rate models were used for this purpose. It was shown what effect have the different models of interest rate on the amount taking into account the different frequency of their payment. Attention has been focused only on net benefits. The determination of the interest rate is of great importance for the correct estimation of the sum of benefits, especially facing the increasing inflation. Let us note that in practice, insurers use various forms of indexation to protect benefits against the effects of inflation. In this area, the use of the fuzzy interest rate in the indexation mechanism may be an interesting and important issue for further research.

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