# Gender and Extended Actuarial Functions in Pension Insurance

**Jana Špirková**[1] | *Matej Bel University, Banská Bystrica, Slovakia*
**Mária Spišiaková** | *Matej Bel University, Banská Bystrica, Slovakia*

***Abstract***

This paper brings analysis of the impact of a ban on the use of gender in insurance, with special stress on pension annuity, according to the requirements of the European Court of Justice. The paper brings a state-of-the-art overview of known and extended actuarial functions which relate to modeling of a premium of endowment, term life insurance and pension annuity. Moreover, the amounts of the pension annuities payable  thly per year in a model of the third pillar pension are modeled and analyzed for different interest rates using life tables for both genders and unisex.

***Keywords***

*Annuity, extension, gender, premium, pension*

***JEL code***

*C 13, G 22*

## INTRODUCTION

The European Union (EU) Gender Directive (Council Directive 2004/113/EC) guarantees equal treatment between men and women in the access to and supply of goods and services. However, the Directive does not prohibit insurers from using gender in the calculation of premiums and benefits, as it contains an exemption to this rule: under Article 5(2), Member States can opt out from banning the use of gender and can allow proportionate differences in insurance premiums and benefits where the use of gender is a determining factor in the assessment of risk based on the relevant and accurate actuarial and statistical data, provided that Member States ensure that such data is compiled, published and regularly updated. All European national legislative assemblies approved the option to use the opt-out for life products – including life insurance and pension annuities. However, on 1st March 2011, the European Court of Justice (ECJ) ruled that this time-unlimited opt-out provision from the EU Gender Directive was inconsistent with the European Charter (Test-Achats ruling). The ECJ ruled that the (time-unlimited) exemption is invalid but allowed for a transition period for implementation up to 21st December 2012.

[1] Faculty of Economics, Matej Bel University, Tajovského 10, 975 90 Banská Bystrica, Slovakia. Corresponding author: e-mail: jana.spirkova@umb.sk, phone: (+421)484466626.

National governments of Member States will be obliged to change their laws accordingly by this date. For more information see also Oxera (2012).

Not only the risk factor – tender, but also actuarial modeling can affect the size of the premium and future size of premiums and pension annuities. The net annual premium of whole life annuities and also other actuarial functions are usually evaluated for integer ages and terms, assuming that cash flows are payable annually. However, annuities are very often paid more frequently than annually, namely monthly, quarterly, but also semi-yearly. We can find in various sources well known formulas for valuation of certain and expected annuities which are paid more frequently than annually, see Booth et al. (1999), Dickson (2009), MacDonald (2012). However, it is possible to extend these actuarial functions, which means to determine them much more accurately.

We introduce much more precise formulas of the expected present value of the annuities payable $m$ times a year on the basis of Woolhouse's formula regarding Maclaurin expansion in MacDonald (2012). With respect to the range of the paper, we only introduce an outline of the derivation. The whole derivation can be found in Špirková, Urbaníková (2012). Moreover, we apply extended actuarial functions on the evaluation of monthly pension annuities in a model of the third pillar pension. Synchronously, we compare the size of monthly pension annuities with respect to life tables of male, female and unisex.

This paper is organized as follows: in the first part we recall two factors which impact the size of premium and also the size of future pension annuities – EU Gender Directive and mathematical model of the expected present values and accumulated future values of annuities payable annually and $m$-thly per year. We present Woolhouse's formula, which was derived according to Maclaurin expansion, see MacDonald (2012). In the second part we develop formulas for $m$-thly annuities payable in advance by the mentioned Woolhouse's formula. In the third part we apply the above-mentioned developed formulas for $m$-thly paid annuities, especially the expected present value of $m$-thly annuity payable in advance, and the accumulated future value of $m$-thly certain annuity payable in advance on the determination of monthly pension annuities in a simplified model of the third pillar pension. At the end, we give some remarks and schemes of our next investigation.

## 1 PRELIMINARIES

Note that in the case of positive cash-flows the present value with payments paid $m$-thly in advance is less than the corresponding yearly present value. In the yearly case, a full payment of the annuity of one monetary unit would be made at time 0. In the $m$-thly case, the annuitant receives $1/m$ of the monetary unit each period of the length $1/m$, and will not have collected the full one amount until one $m$-th of a year before the year end. This is true for each year of the annuity. There is therefore a loss of interest, which is reflected in a lower present value for the $m$-thly case.

Exact calculations of Euler-Maclaurin expansion are not always used in practice. It is common to use approximations. A one-year, one-unit annuity payable $m$-thly is simply an annuity paying $1/m$ units for $m$ periods. We will refer to these as $m$-thly annuities or annuities payable $m$-thly. In this way we can derive the corresponding results for the expected present value of an $m$-thly paid annuity. We consider here a whole annuity-due of one monetary unit per annum, payable to a life age $x$ with $1/m$ payable at the beginning of each $m$-thly period. Consider a whole life annuity with payments of $1/m$ made $m$ times a year at moments $0, 1/m, \ldots, (m-1)/m$. So, for each complete year, the annual premium is one. Say, if $m = 12$ , an $m$-th is a month.

Proposition 1 uses Woolhouse's formula for annuities in advance which can be derived according to Maclaurin expansion. Regarding the range of our paper, we do not provide the whole derivation of Woolhouse's formula, but it can be found for example in Dickson (2009).

*Proposition 1 – Woolhouse's formula*
Let *f(x)* be a continuous function on an interval $[a, b]$where $h = \frac{b-a}{N}$ is the width of individual intervals and $N$ is the number of divisions of the interval $[a, b]$, $m$ is the number of divisions of the interval of the width $h$, $1<m$, $m \in Z$.
Then the approximation:

$$\frac{1}{m} f(a) + \frac{1}{m} f\left(a + \frac{h}{m}\right) + \cdots + \frac{1}{m} f\left(a + \frac{(Nm-1)h}{m}\right) \approx$$
$$\approx f(a) + f(a+h) + \cdots + f(a + (N-1)h) - \frac{m-1}{2m}\,(f(a) - f(a+Nh)) + \quad (1)$$
$$+ \frac{h}{12} \times \frac{m^2-1}{m^2}\,(f'(a) - f'(a+Nh)) - \frac{h^3}{720} \times \frac{m^4-1}{m^4}\,(f'''(a) - f'''(a+Nh))$$

holds.

If we want to derive the present value of $m$-thly paid annuities in the size of $1/m$ of monetary unit according to (1) the function $f$ is represented by $f(t) = {}_tp_x \times v^t$ as a function of time $t$, where ${}_tp_x$ is the probability that $x$-aged man will be alive at age $x + t$ and $v = 1/(1 + i)$ is a discounting factor, $i$ is a technical interest rate. If we do not assume terms with derivatives, then the value $f(a)$ at (1) for $t = 0$ is 1, and the value $f(a + Nh)$ for $t \to \infty$ tends to 0. Seeing that the left side of equation (1) represents the expected present value of the whole annuity of one per year, payable to the entry aged $x$ with $1/m$ at the beginning of each $m$-thly period, we get the approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}, \quad (2)$$

where $\ddot{a}_x$ is the present value of whole life yearly annuities in the amount of 1 monetary unit, $\ddot{a}_x = \sum_{t=0}^{\omega - x} {}_tp_x \times v^t$ or by the commutation functions $\ddot{a}_x = \frac{N_x}{D_x}$. Commutation function $D_x = l_x \times v^x$ represents the discounted number of survivors at age $x$, and $N_x = \sum_{i=0}^{\omega - x} D_{x+i}$.

The theory about commutation functions can be found, for example, in Gerber (1997), Urbaníková, Vaculíková (2006).

## 2 EXTENDED FORMULAS OF ANNUITIES

If we would like to extend the approximation (2) with at least one term with derivative from (1), we can do the next reflection: we assume the force of mortality $\mu_x$ at time $t$ constant and then the probability that $x$-age insured group will be alive at age $x + t$ is given by:

$${}_tp_x = e^{-\mu_x t}. \quad (3)$$

The value $\mu_x$ is known as the force of mortality at age $x$ and has several equivalent forms, see also Promislow (2006), Gerber (1997). The quantity $\mu_x$ gives us the "relative rate" of decline in this group at age $x$.

For example, from (3) we can see that the force of mortality at age $x$ can be rewritten by:

$$\mu_x = -\frac{\frac{d\,{}_tp_x}{dt}}{{}_tp_x}. \quad (4)$$

We can view $\mu_x$ as the force of mortality at time $t$ for an individual age $x$ and approximate by $\mu_x \approx -\ln p_x$.

The value $\delta = \ln(1 + i)$ represents the so-called force of interest. From the derivative of the function ${}_tp_x \times v^t$ according to $t$ we get the expression below:

$$\frac{d}{dt}\,{}_tp_x \times v^t = -\,(\mu_x + \delta) \times {}_tp_x \times v^t. \tag{5}$$

Regarding the previous consideration we get a more precise approximation of the expression (2), which is given by:

$$\ddot{a}_x^{(m)*} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2} \times (\mu_x + \delta). \tag{6}$$

Although the values $\mu_x$, $\delta$ are independent, for $t \to 0$ we can approximate the mentioned sum by $2i$ (according to their average values and also for simplicity), we can rewrite the previous formula by:

$$\ddot{a}_x^{(m)**} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{6m^2} \times i\,. \tag{7}$$

Similarly, we can derive an expression for the present value of deferred whole life annuity in advance ${}_{k|}\ddot{a}_x^{(m)}$ which is paid $m$-thly per year in the amount $1/m$ of monetary unit from age $x + k$. It is given by:

$${}_{k|}\ddot{a}_x^{(m)} \approx {}_{k|}\ddot{a} - \frac{m-1}{2m}\,\frac{D_{x+k}}{D_x}\,. \tag{8}$$

An extended version of the previous formula is as follows:

$${}_{k|}\ddot{a}_x^{(m)*} \approx {}_{k|}\ddot{a}_x - \frac{D_{x+k}}{D_x} \times \left(\frac{m-1}{2m} + \frac{m^2-1}{12m^2} \times (\mu_x + \delta)\right), \tag{9}$$

or in a simplified form:

$${}_{k|}\ddot{a}_x^{(m)**} \approx {}_{k|}\ddot{a}_x - \frac{D_{x+k}}{D_x} \times \left(\frac{m-1}{2m} + \frac{m^2-1}{6m^2} \times i\right). \tag{10}$$

For the purposes of our model we recall the accumulated value of temporary $m$-thly paid certain annuities of $1/m$ monetary unit which are payable in advance during $n$ years, which is denoted as $\ddot{s}_{\overline{n}|}^{(m)}$ and is as follows:

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{1}{m} \times q^{\frac{1}{m}} \times \frac{q^n - 1}{q^{\frac{1}{m}} - 1}\,. \tag{11}$$

For more information see also Sekerová, Bilíková (2007), Urbaníková (2008).

*Remark 1*

According to Woolhouse`s formula we can derive approximation of formula (11) according to $f(t) = (1 + i)^{n-t}$. Our model represents a deterministic approach in pension annuities. Stochastic models are described, for example in Booth et al. (1999), Gerber (1997), Potocký et al. (2007), Potocký (2008).

## 3 APPLICATION OF EXTENDED ACTUARIAL FUNCTIONS

In the previous part we introduced the extension of some basic formulas for expected life annuities. Now, we will apply the mentioned formulas for determination and analysis of pension annuities from a simplified model of the third pillar pension, which represents a whole life annuity payable monthly in advance. Of course, we know that many factors such as raising of wage, inflation, costs of Pension Asset Management Companies, costs of insurance companies, etc., will influence the future pension. But we would like to emphasize that in our model we can show how the risk factor gender and extended actuarial functions, namely the present value of $m$-thly paid annuities $\ddot{a}_x^{(m)}$, and its extensions $\ddot{a}_x^{(m)**}$ or $\ddot{a}_x^{(m)*}$, influence the size of future pension annuities. In our model we evaluate the accumulated value of certain annuities in advance (11) as future accumulated value of funds, from which pension annuities will be paid out. We introduce basic formulas for determination of monthly annuities from the third pillar pension.

With regard to (2) and (11), monthly pension annuities can be determined as follows:

$$p^{(m)} = \frac{A \times \ddot{s}_{\overline{n}|}^{(m)} \times (1-p)}{m \times \ddot{a}_x^{(m)}}, \tag{12}$$

where $A$ is a monthly payment of annuities during $n$ years, $A \times \ddot{s}_{\overline{n}|}^{(m)}$ represents the real accumulated value over duration time $n$, $p$ represents the first higher pension (in percent).

For more precise evaluation we can apply a precise formula according to (6), (11) which is given by:

$$p^{(m)*} = \frac{A \times \ddot{s}_{\overline{n}|}^{(m)} \times (1-p)}{m \times \ddot{a}_x^{(m)*}}. \tag{13}$$

We can express the previous formula according to (7) and (11) as follows:

$$p^{(m)**} = \frac{A \times \ddot{s}_{\overline{n}|}^{(m)} \times (1-p)}{m \times \ddot{a}_x^{(m)**}}. \tag{14}$$

*Remark 2*
$m$-thly paid pension annuities can be modified according to the requirements of clients. For example, they can be expressed as follows:

$$p^{(m)\circ} = \frac{A \times \ddot{s}_{\overline{n}|}^{(m)} \times (1-p)}{m \times (\ddot{a}^{(m)*} + z \times \ddot{a}_{\overline{n}|}^{(m)} A_x)}, \tag{15}$$

where $z$ is the coefficient of the survivor´s pension arrangements, $\ddot{a}_{\overline{n}|}^{(m)}$ represents the present value of temporary $m$-thly paid certain annuities of $1/m$ monetrary unit which are payable in advance during $n$ years and is given by $\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m} \times q^{\frac{1}{m}} \times \frac{1-q^{-n}}{q^{\frac{1}{m}}-1}$, the value $A_x$ is the net single premium of whole life insurance for $x$-aged client with the policy value of one monetary unit. It can be expressed by commutation functions as follows $A_x = \frac{M_x}{D_x}$, where $M_x = \sum_{i=0}^{\omega - x} C_{x+i}$ and $C_x = d_x \times v^{x+½}$ is the discounted number of deaths at age $x$. The whole evaluation is based on life tables from Mortality tables (2012).

In our model we assume that a client at the age of 18 saves monthly 30 euros in a pension company during the whole duration time until retirement. The first higher pension is 10% from accumulated value.

Firstly, we recall that $m$-thly pensions according to formulas (13)–(15) are not significantly different for small technical interest rates. For smaller technical interest rates around the recent interest rate $i$ = 2.5% p.a. the difference between using known and extended formulas for pension from the third pillar pension is only a few cents. If we assume a technical interest rate $i$ = 7.5% p.a., the values differ in a few euros.

In Table 1 are evaluated monthly annuities from our model with entry age to retirement.

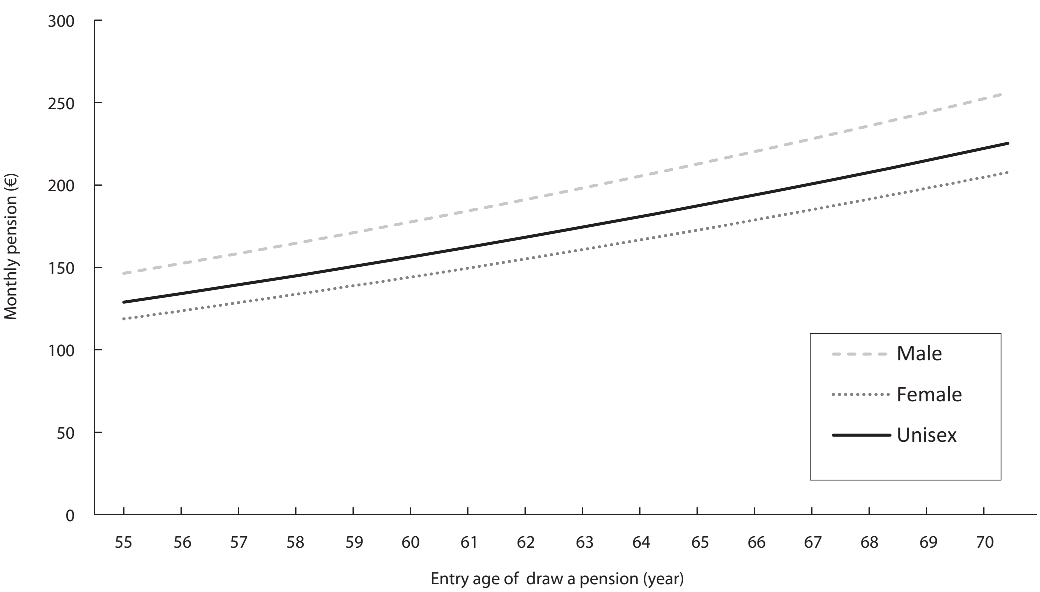
**Table 1** Monthly annuities (in euros) for client from our model according to formula (12)

| | *i* = 2.5% p.a. | | | *i* = 7.5% p.a. | | |
|---|---|---|---|---|---|---|
| x | male | female | unisex | male | female | unisex |
| 55 | 146.31 | 118.76 | 128.83 | 639.16 | 544.41 | 580.70 |
| 56 | 152.46 | 123.72 | 134.20 | 690.64 | 588.26 | 627.48 |
| 57 | 158.68 | 128.80 | 139.72 | 745.98 | 635.40 | 677.76 |
| 58 | 165.09 | 134.01 | 145.36 | 805.48 | 686.07 | 731.81 |
| 59 | 171.67 | 139.35 | 151.16 | 869.43 | 740.55 | 789.91 |
| 60 | 178.41 | 144.82 | 157.09 | 938.18 | 799.11 | 852.38 |
| 61 | 185.32 | 150.43 | 163.17 | 1 012.09 | 862.06 | 919.53 |
| 62 | 192.40 | 156.18 | 169.41 | 1 091.54 | 929.73 | 991.71 |
| 63 | 199.66 | 162.07 | 175.80 | 1 176.95 | 1 002.48 | 1 069.31 |
| 64 | 207.10 | 168.11 | 182.35 | 1 268.77 | 1 080.68 | 1 152.73 |
| 65 | 214.73 | 174.30 | 189.07 | 1 367.47 | 1 164.76 | 1 242.40 |
| 66 | 222.55 | 180.65 | 195.95 | 1 473.57 | 1 255.13 | 1 338.80 |
| 67 | 230.56 | 187.15 | 203.01 | 1 587.64 | 1 352.28 | 1 442.43 |
| 68 | 238.77 | 193.82 | 210.24 | 1 710.25 | 1 456.72 | 1 553.84 |
| 69 | 247.19 | 200.65 | 217.65 | 1 842.07 | 1 569.00 | 1 673.60 |
| 70 | 255.82 | 207.66 | 225.25 | 1 983.77 | 1 689.69 | 1 802.33 |

**Source:** Own construction

In Figure 1 we can see the dependence of the size of monthly pension according to entry age separately for male, female and unisex according to (13) and corresponding to life tables, which were determined with respect to a technical interest rate $i$ = 2.5% p.a. Based on the data available for the Slovak Republic with the technical interest rate 2.5% p.a. men (aged 55 and more) could see a reduction in pension income from pension annuities of around 12% or more on average; women (aged 55 and more) could see pension income rise of around 8% or more on average, see Table 1. Moreover, additional costs could arise from insurers applying a gender mix risk premium due to the risk of adverse selection. There could also be additional marketing costs due to a ban on the use of gender.

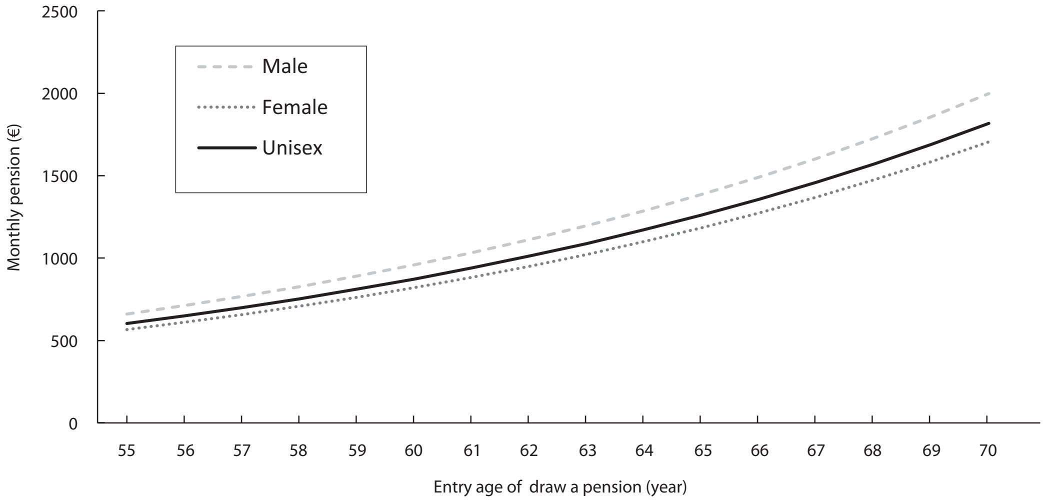
**Figure 1** Dependence of the size of monthly pension according to entry age (technical interest rate 2.5% p.a.)




**Source:** Own construction

With a technical interest rate 7.5% p.a. men (aged 55 and more) could see a reduction in pension income from pension annuities of around 9% or more on average; women (aged 55 and more) could see pension income rise of around 7% or more on average, see Table 1 and Figure 2.

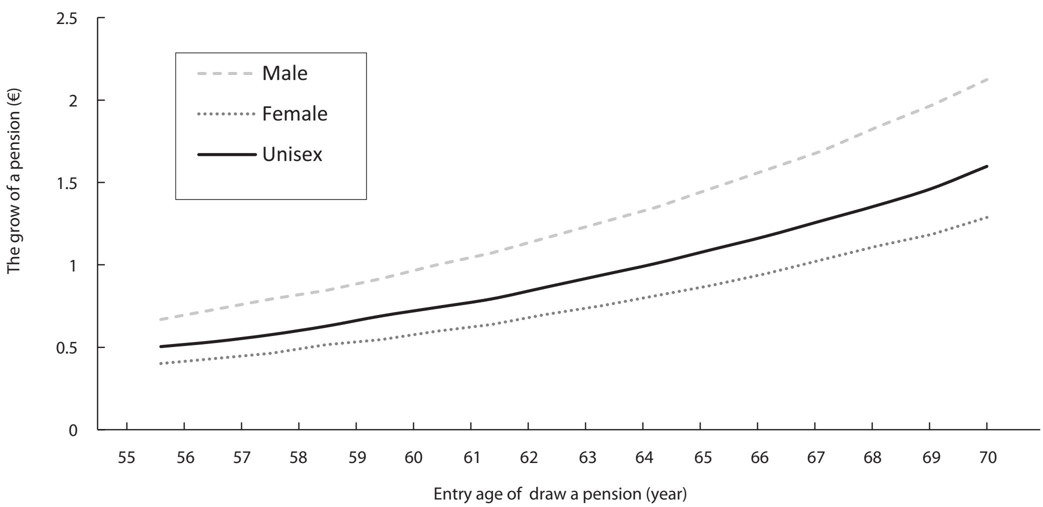
**Figure 2** Dependence of the size of monthly pension according to entry age (technical interest rate 7.5% p.a.)




**Source:** Own construction

In Figure 3 is shown the difference in the size of pension annuity based on the comparison of formulas (12) and (13) regarding the interest rate 7.5% p.a.

**Figure 3** The difference in the size of the pension with respect to formulas (12) and (13) according to entry age (technical interest rate 7.5% p.a.)




**Source:** Own construction

All values and graphs were evaluated and constructed by MS Office Excel 2010 system.

## CONCLUSION

In our paper we discuss the size of pension annuities with respect to the rule of ECJ and well-known and extended formulas for -thly paid annuities.

For annuities, women receive a lower pension annuity payment monthly than men for the same accumulated value in the time of retirement. However, women have a higher life expectancy, which means that women receive pension annuities over a longer time, and so women receive the same expected lifetime annuity benefit as men. Acording to Oxera (2012) focusing on the analysis of the gender as a risk factor in insurance in selected European countries, namely Germany, France, Spain, Poland, Czech Republic and Belgium, men could see a reduction in pension income from pension annuities of around 5% or more on average. However, this source does not study the dependence on the interest rate.

Based on the data available for the Slovak Republic with the technical interest rate 2.5% p.a. men could see a reduction in pension income from pension annuities of around 12% or more on average and women could see a pension income rise of around 8% or more on average. With a technical interest rate 7.5% p.a. a reduction for men is around 9% or more and a pension income rise of around 7% or more on average for women.

From a mathematical point of view it is interesting to extend the mentioned formulas. But their application is useful only for a very long duration of time and higher monthly payments and higher technical interest rates. Moreover, the mentioned extended formulas are suitable for modeling of second and third pillar pensions and also deferred pension annuities.

## ACKNOWLEDGEMENTS

This work was supported by the Civic Association ECONOMY.

## *References*

BOOTH, P. et al. *Modern Actuarial Theory and Practice*. Chapman & Hall / CRC, 1999.

DICKSON, D. C. M. et al. *Actuarial Mathematics for Life Contingent Risks*. New York: Cambridge University Press, 2009.

GERBER, H. U. et al. *Life Insurance Mathematics*. Berlin, Heidelberg, New York : Springer-Verlag, 1997.

MACDONALD, A. S. *Euler-Maclaurine Expansion and Woolhouse's Formula* [online]. [cit. 3.6.2011]. <http://perso.univ-rennes1.fr/arthur.charpentier/euleract.pdf>.

MORTALITY TABLES. *Mortality tables* [online]. [cit. 10.3.2012]. <http://portal.statistics.sk/showdoc.do?docid=2464>.

OXERA. *The Impact of a Ban on the Use of Gender in Insurance* [online]. [cit. 14.3.2012]. <http://www.slaspo.sk/tmp/asset_cache/link/0000033890/11207%20oxera-study-on-gender-use-in-insurance.pdf>.

POTOCKÝ, R. On a Dividend Strategy of Insurance Companies. *Ekonomie a Management*, Vol. 4, 2008, pp. 103–109.

POTOCKÝ, R., STEHLÍK, M. *Stochastic Models in Insurance and Finance with Respect to Basel II. JAMSI*. 2007, No. 2, Vol. 3, pp. 237–245. ISSN 1336-9180.

PROMISLOW, D. *Fundamentals of Actuarial Mathematics*. Toronto, Canada: John Wiley & Sons, 2006.

SEKEROVÁ, V., BILÍKOVÁ, M. *Insurance Mathematics* (in Slovak). Bratislava: Ekonóm, 2007.

ŠPIRKOVÁ, J., URBANÍKOVÁ, M. *Actuarial Mathematics – Life Insurance* (in Slovak). Bratislava: Iura Edition, spol s r.o., 2012.

URBANÍKOVÁ, M. *Financial Mathematics* (in Slovak). Nitra: Constantine The Philosopher University, 2008.

URBANÍKOVÁ, M.,VACULÍKOVÁ, Ľ. *Actuarial Mathematics* (in Slovak). Bratislava: Slovak University of Technology, 2006.