Evolution of the GEKS index

Jacek Białek[1] | *University of Lodz, Łódź, Poland*

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Abstract

The GEKS index is a well-known multilateral index used by many statisticians to measure inflation based on scanner data. As a rule, it is assumed that the underlying index within the GEKS formula satisfies the time reversal test. Most often, this underlying index is assumed to be superlative, and therefore, the GEKS index based on the Fisher, Törnqvist (GEKS-T or CCDI) or Walsh (GEKS-W) formulas are most often considered. However, the 'classic' GEKS index does not meet the stringent identity test. Unfortunately, most of the known multilateral indices do not meet the time reversal test, which many researchers consider a weakness. This paper reviews both well-known and lesser-known modifications and generalizations of the GEKS index. We discuss three new and general classes of indices based on the GEKS method, as well as some special cases of these classes, which include the GEKS, GEKS-T, and GEKS-W indices and, additionally, the GEKS-L and GEKS-GL indices. One class uses elasticity of substitution, so it is close to the economic approach, while another class - like the multilateral Geary-Khamis index- uses quality-adjusted prices and quantities. Not all the GEKS modifications discussed require the underlying index to meet the time reversal test. It should be noted that in cases where this assumption has been dropped, an identity test has been gained. We present the basic axiomatic properties of the proposed indices and compare them empirically based on real and available scanner datasets. The simulation study verifies the impact of price and quantity volatilities on the differences between the price indices discussed.

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inflation measurement, Consumer Price Index (CPI), scanner data, multilateral indices, identity test, GEKS method	*https://doi.org/10.54694/stat.2025.58*	*C43, E52*

INTRODUCTION

Among the multitude of challenges facing statistical offices that use scanner data to measure inflation, the choice of an appropriate price index formula is a key issue (see ABS (2016), CPI Manual (2004). In general, bilateral unweighted and weighted price indices, chain indices, and multilateral indices can be considered for this purpose. However, the choice of price index should lead to a reduction in both substitution bias and chain drift bias (Chessa, Verburg, and Willenborg, 2017). It is important to note that in literature, there is a set of criteria that can be taken into account when choosing a price index formula. These include the *axiomatic*, *economic* and *stochastic approaches* (Eurostat, 2022), as well as the complementary *time-consuming approach* and the *new stochastic approach* (Białek and Beręsewicz, 2021). The consensus among statisticians is that for scanner data, the most rational choice of price index

[1] Department of Statistical Methods, University of Lodz, Łódź POW 3/5 90-255 Łódź, Poland. E-mail: jacek.bialek@uni.lodz.pl, Statistics Poland, Department of Price and Services. E-mail: J.Bialek@stat.gov.pl.
ORCID: <https://orcid.org/0000-0002-0952-5327>.

is a multilateral index, which operates on a time window and thus takes into account the high turnover of scanned products (see Eurostat (2022) for details).

It is well known that GEKS-type indices (e.g., the GEKS, CCDI, and GEKS-W indices) have good axiomatic properties and are close to the *economic approach* since they are based on superlative indices. The GEKS index has been shown (Gini, 1931; Eltetö and Köves, 1964) to be exact for a "flexible" functional form, i.e., it expresses the price differences experienced by optimizing consumers without imposing restrictive assumptions about how they can substitute between products. Moreover, GEKS-type indices are relatively fast in computations, which can be of practical importance when dealing with large scanner datasets (Białek, 2022b). This paper reviews more or less well-known modifications and generalizations of the GEKS index. We discuss several general classes of indices based on the idea of the GEKS method and some special cases of these classes, which include GEKS, GEKS-T and GEKS-W indices and additionally GEKS-L and GEKS-GL indices. One class uses elasticity of substitution, so it is close to the economic approach, while another class - like the multilateral Geary-Khamis index - uses quality-adjusted prices and quantities. Not all the GEKS modifications discussed require the underlying index to meet the time reversal test. Note that where this assumption has been dropped, an identity test has been gained (see Theorems 1, 2, and 5). We present the basic axiomatic properties of the proposed indices, and in an empirical study, we compare them based on real scanner datasets. The simulation study verifies the impact of price and quantity volatilities on the differences between the price indices discussed.

It should be noted that GEKS-type indices are not the only multilateral indices designed for scanner data. For example, the Time-Product Dummy (TPD) index is based on the regression framework discussed in De Haan and Krsinich (2014), who analyzed scanner data and addressed the treatment of quality change in non-revisable price indexes using time and product dummy variables. Nevertheless, a discussion of other classes of multilateral price indices is beyond the scope of this study.

1 THE GEKS-TYPE INDEX IDEA

Let us denote sets of homogeneous products belonging to the same product group in the months 0 and t by G_0 and G_t, respectively, and let $G_{0,t}$ denote a set of matched products in both moments 0 and t. Let p_i^τ and q_i^τ denote the price and quantity of the i-th product at the time τ and let $N_{0,t}$ be the number of elements of set $G_{0,t}$.

The GEKS price index between the months 0 (the base period) and t (the current period) is an un--weighted geometric mean of the $T + 1$ ratios of bilateral price indices $P^{\tau,t}$ and $P^{\tau,0}$, which are based on the same price index formula. Typically, the GEKS method uses the superlative Fisher price index (Fisher 1922), resulting in the following formula:

$$P_{GEKS}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_F^{\tau,t}}{P_F^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{1}$$

where $P_F^{\tau,t}$ denotes the Fisher price index calculated for products from the set $G_{\tau,t}$.

The GEKS method for making international index number comparisons was proposed by (Gini, 1931), although it was derived in a different manner by Eltetö and Köves (1964) and Szulc (1964). The CCDI price index, based on the Törnqvist (1936) index, can be expressed as follows:

$$P_{CCDI}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_T^{\tau,t}}{P_T^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{2}$$

where $P_T^{\tau,t}$ denotes the Törnqvist price index calculated for products from the set $G_{\tau,t}$.

In Chessa, Verburg, and Willenborg (2017), a hint is provided for selecting a base index formula for the GEKS method: "the bilateral indices should satisfy the time reversal test." However, it is most often assumed that the price index formula found in the body of the GEKS index is a superlative formula (van Loon and Roels, 2018; Diewert and Fox, 2018). For this reason, a GEKS index based on the superlative Walsh (1901) index, which we will refer to as GEKS-W, is also often considered. It can be written as follows:

$$P_{GEKS-W}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_W^{\tau,t}}{P_W^{\tau,0}} \right)^{\frac{1}{T+1}}, \tag{3}$$

where $P_W^{\tau,t}$ denotes the Walsh price index calculated for products from the set $G_{\tau,t}$.

2 SEMI GEKS INDICES

New GEKS-type price indices have recently appeared in the literature that challenge traditional assumptions associated with the underlying index found in the body of the GEKS index. These novel GEKS-type indices are based neither on a superlative price index nor on an index that meets the *time reversal test*. In Białek (2022b) and Białek (2022a), a general class of such indices (denoted by GS-GEKS) is proposed, and its two special cases, i.e., the GEKS-L and GEKS-GL index, are discussed (see Sections 3.1 and 3.2). Without elaborating further here, the papers cited demonstrate that under appropriate assumptions for the base formula, indices from the general class of semi-GEKS methods (GS-GEKS) satisfy many valuable tests (including the *identity test*).

2.1. The GEKS-L index

Let $P_L^{\tau,s}$ denote the Laspeyres (1871) price index, which compares period s to period τ using data from $G_{\tau,s}$. This index is defined as:

$$P_L^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^\tau}. \tag{4}$$

Under this notation, the GEKS-L index can be defined as follows (Białek, 2022a):

$$P_{GEKS-L}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_L^{\tau,t}}{P_L^{\tau,0}} \right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} \left(\frac{\dfrac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^\tau}}{\dfrac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{5}$$

The GEKS-L index can be treated as a generalization of the Fisher price index formula ($P_F^{0,t}$) to a multi-period case. Specifically, in a static item universe G observed over the two-period time interval [0,1], we obtain:

$$P_{GEKS-L}^{0,1} = \prod_{\tau=0}^{1} \left(\frac{\sum_{i \in G} q_i^\tau p_i^1}{\sum_{i \in G} q_i^\tau p_i^0} \right)^{\frac{1}{1+1}} = \left(\frac{\sum_{i \in G} q_i^0 p_i^1}{\sum_{i \in G} q_i^0 p_i^0} \times \frac{\sum_{i \in G} q_i^1 p_i^1}{\sum_{i \in G} q_i^1 p_i^0} \right)^{\frac{1}{2}} = P_F^{0,1}, \tag{6}$$

since $G_0 = G_1 = G_{0,1} = G$.

2.2. The GEKS-GL index

Let $P_{GL}^{\tau,s}$ denote the geometric Laspeyres price index formula, which compares period s to period τ using data from $G_{\tau,s}$. This index is defined as:

$$P_{GL}^{\tau,s} = \prod_{i \in G_{\tau,s}} \left(\frac{p_i^s}{p_i^\tau} \right)^{w_i^{\tau,s}(\tau)}, \tag{7}$$

where

$$w_i^{\tau,s}(\tau) = \frac{q_i^\tau p_i^\tau}{\sum_{k \in G_{\tau,s}} q_k^\tau p_k^\tau}. \tag{8}$$

Using this notation, the GEKS-GL index can be defined as follows (Białek, 2022a):

$$P_{GEKS-GL}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{P_{GL}^{\tau,t}}{P_{GL}^{\tau,0}} \right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T} \left(\frac{\prod_{i \in G_{\tau,t}} \left(\frac{p_i^t}{p_i^\tau} \right)^{w_i^{\tau,t}(\tau)}}{\prod_{i \in G_{\tau,0}} \left(\frac{p_i^0}{p_i^\tau} \right)^{w_i^{\tau,0}(\tau)}} \right)^{\frac{1}{T+1}}. \tag{9}$$

It is noteworthy that the GEKS-GL index can be treated as a generalization of the Törnqvist price index ($P_T^{0,t}$) to a multi-period case. Specifically, in a static item universe observed over the two-period time interval [0,1], we obtain:

$$P_{GEKS-GL}^{0,1} = P_T^{0,1}. \tag{10}$$

since $G_0 = G_1 = G_{0,1} = G$, and consequently $w_i^{\tau,0}(\tau) = w_i^{\tau,1}(\tau) = w_i(\tau)$ for any τ.

Białek (2022a) proves the following theorem:

Theorem 1. *The GEKS-L and GEKS-GL indices satisfy the following tests: transitivity, identity, multi-period identity, responsiveness, continuity, positivity and normalization, commensurability, price proportionality, homogeneity in prices, and homogeneity in quantities.*

3 QUALITY-ADJUSTED GEKS-TYPE INDICES

Two new multilateral indices, which superficially resemble the GEKS index, were proposed in Białek (2023) (building upon discussions in Białek (2022b) and Białek and Pawelec (2025)). However, their base index structure differs completely from the established convention of using superlative indices. Moreover, calculating the base index requires *quality adjusting*, which, in turn, is more like the Geary-Khamis index idea. These proposed indices represent a hybrid approach, bridging the gap between quality-adjusted unit value methods and the GEKS method.

3.1. The GEKS-AQU index

In the unit value concept, prices of homogeneous products are equal to the ratio of expenditure to quantity sold (CPI Manual, 2004; Chessa, Verburg, and Willenborg, 2017). However, unlike homogeneous products, quantities of different products cannot be added together. That is why the idea of quality-adjusted unit value assumes that prices p_i^s of different products $i \in G_s$ in month s are transformed into "quality-adjusted prices" $\frac{p_i^s}{v_i}$, and quantities q_i^s are converted into "common units" $v_i q_i^s$ by using a set of factors $v = \{v_i : i \in G_s\}$ (Chessa, Verburg, and Willenborg, 2017). Thus, the "classical" quality-adjusted unit value $QUV_{G_s}^s$ of a set of products G_s in month s can be expressed as follows:

$$QUV_{G_s}^s = \frac{\sum_{i \in G_s} q_i^s p_i^s}{\sum_{i \in G_s} v_i q_i^s}. \tag{11}$$

The term "Quality-adjusted unit value method" (QU method for short) was introduced by Chessa (2015). The QU method is a family of unit value-based index methods, and its general form can be expressed by the following ratio:

$$P_{QU}^{0,t} = \frac{QUV_{G_t}^t}{QUV_{G_0}^0}. \tag{12}$$

In practice, consumer responses to price changes can be delayed or even accelerated, as consumers not only react to current price changes but also use their own "forecasts" or concerns about future price increases. For example, the consumption of thermophilic (seasonal) fruit is likely to be higher in summer because they are cheaper than in winter, when the season is almost over. For instance, some interesting studies on "unconventional" consumer behavior, such as stocking and delayed quantity responses to price changes, and their impact on chain drift bias, can be found in (von Auer, 2019). Given that, in practice, prices and quantities are often not perfectly synchronized in time, the following form of the "asynchronous quality-adjusted unit value" is proposed:

$$AQUV_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i \in G_{\tau,s}} q_i^\tau p_i^s}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}, \tag{13}$$

where τ is any period from the considered time interval $[0,T]$. It is thus evident that $AQUV_{G_{s,s}}^{s,s} = QUV_{G_s}^s$. Let us now define the function $P^{\tau,s}\left(v, q^\tau, p^\tau, p^s\right)$ as follows:

$$P^{\tau,s}\left(v, q^\tau, p^\tau, p^s\right) = \frac{AQUV_{G_{\tau,s}}^{\tau,s}}{AQUV_{G_{\tau,\tau}}^{\tau,\tau}}. \tag{14}$$

Substituting (14) into the GEKS formula, we obtain:

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\dfrac{\sum_{i \in G_{\tau,t}} q_i^\tau p_i^t}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}{\dfrac{\sum_{i \in G_{\tau,0}} q_i^\tau p_i^0}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}} \right)^{\frac{1}{T+1}}. \tag{15}$$

It is important to note that the proposed index behaves like a GEKS index based on the Laspeyres index when the item universe G is static. Specifically, if the item universe is static, we obtain

$$P_{GEKS-AQU}^{0,t} = \prod_{\tau=0}^{T}\left(\frac{\dfrac{\sum_{i\in G} q_i^{\tau} p_i^{t}}{\sum_{i\in G} v_i q_i^{\tau}}}{\dfrac{\sum_{i\in G} q_i^{\tau} p_i^{0}}{\sum_{i\in G} v_i q_i^{\tau}}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\sum_{i\in G} q_i^{\tau} p_i^{t}}{\sum_{i\in G} q_i^{\tau} p_i^{0}}\right)^{\frac{1}{T+1}} = \prod_{\tau=0}^{T}\left(\frac{\dfrac{\sum_{i\in G} q_i^{\tau} p_i^{t}}{\sum_{i\in G} q_i^{\tau} p_i^{\tau}}}{\dfrac{\sum_{i\in G} q_i^{\tau} p_i^{0}}{\sum_{i\in G} q_i^{\tau} p_i^{\tau}}}\right)^{\frac{1}{T+1}} = P_{GEKS-L}^{0,t}. \tag{16}$$

Finally, it should also be noted that the class of indices is theoretically infinite since different choices of v_i factors lead to different index values. In this paper, we adopt the system of weights v_i that correspond to the augmented Lehr index (Lamboray, 2017; van Loon and Roels, 2018), where

$$v_i = \frac{\sum_{t=0}^{T} p_i^t q_i^t}{\sum_{t=0}^{T} q_i^t}. \tag{17}$$

3.2. The GEKS-AQI index

Formula (13) can be expressed by using quality-adjusted prices and quantities:

$$AQUV_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau} \dfrac{p_i^s}{v_i}}{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau}}. \tag{18}$$

If we replace all the adjusted prices ($\frac{p_i^s}{v_i}$) with the relative prices ($\frac{p_i^s}{p_i^{\tau}}$), then we obtain an "asynchronous quality-adjusted price index" (AQI), i.e.,

$$AQI_{G_{\tau,s}}^{\tau,s} = \frac{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau} \dfrac{p_i^s}{p_i^{\tau}}}{\sum_{i\in G_{\tau,s}} v_i q_i^{\tau}}. \tag{19}$$

This means that the AQI formula can be treated as a weighted arithmetic mean of partial indices $\frac{p_i^s}{p_i^{\tau}}$, where the weights are proportional to the relative share of the product's adjusted quantities (from the base period τ) in the sum of all adjusted quantities.

After inserting (19) into the GEKS formula, we obtain the following:

$$P_{GEKS-AQI}^{0,t} = \prod_{\tau=0}^{T}\left(\frac{\dfrac{\sum_{i\in G_{\tau,t}} v_i q_i^{\tau} \dfrac{p_i^t}{p_i^{\tau}}}{\sum_{i\in G_{\tau,t}} v_i q_i^{\tau}}}{\dfrac{\sum_{i\in G_{\tau,0}} v_i q_i^{\tau} \dfrac{p_i^0}{p_i^{\tau}}}{\sum_{i\in G_{\tau,0}} v_i q_i^{\tau}}}\right)^{\frac{1}{T+1}}. \tag{20}$$

Note that while the GEKS-AQI index takes into account prices and quantities directly from all-time window periods, the GEKS-AQU index takes into account all quantities but only prices from the reference and base periods. However, both formulas indirectly need information about the prices (and quantities) of products from each period in the time window to determine the factors v_i defined by formula (17). In this way, each new product in the analyzed time window has an impact on the final value of the proposed indices.

3.3. The GEKS-GAQI index

Please note that the AQI defined in (19) can be viewed as a weighted version of the Carli (1804) price index. Currently, there is a trend in price statistics toward shifting from the Carli index to the Jevons (1865) index, which replaces the arithmetic mean of the sub-indices with the geometric mean. This shift serves as the foundation for a definition of *the geometric version of the asynchronous quality-adjusted price index* (GAQI), which was proposed by Białek and Pawelec (2025). The GAQI price index can be expressed as follows:

$$GAQI_{G_{\tau,s}}^{\tau,s} = \prod_{i \in G_{\tau,s}} \left(\frac{p_i^s}{p_i^\tau} \right)^{\frac{v_i q_i^\tau}{\sum_{i \in G_{\tau,s}} v_i q_i^\tau}} . \tag{21}$$

On the basis of (21), the following form of the GEKS-GAQI multilateral price index was proposed (Białek and Pawelec, 2025):

$$P_{GEKS-GAQI}^{0,t} = \prod_{\tau=0}^{T} \left(\frac{\prod_{i \in G_{\tau,s}} \left(\frac{p_i^t}{p_i^\tau} \right)^{\frac{v_i q_i^\tau}{\sum_{i \in G_{\tau,t}} v_i q_i^\tau}}}{\prod_{i \in G_{\tau,s}} \left(\frac{p_i^0}{p_i^\tau} \right)^{\frac{v_i q_i^\tau}{\sum_{i \in G_{\tau,0}} v_i q_i^\tau}}} \right)^{\frac{1}{T+1}} . \tag{22}$$

The following theorem can be proved (see, for instance, Białek (2023) or Białek and Pawelec (2025)):

Theorem 2. *The GEKS-AQU, GEKS-AQI, and GEKS-GAQI indices satisfy the following tests: transitivity, identity, multi-period identity, responsiveness, continuity, positivity and normalization, price proportionality, and weak commensurability. If the item universe is the same in the compared periods and , then the GEKS-AQU, GEKS-AQI, and GEKS-GAQI indices also satisfy the tests of homogeneity in prices and homogeneity in quantities.*

4 GENERALIZATIONS OF THE GEKS METHOD

Sections 5.1, 5.2, and 5.3 present proposals for certain generalizations of the GEKS index whose constructs originate from the economic approach. Therefore, these proposals will be preceded by a brief introduction that contains basic concepts and definitions related to the Cost of Living Index (COLI). Considerations regarding the following general classes of indices can be found in Białek (2025).

4.1. The Cost of Living Index (COLI)

The theory of the Cost of Living Index (COLI) for a single consumer (or household) was first developed by Konüs (1924). In this economic approach, the period τ quantity vector $q^\tau = \left[q_1^\tau, q_2^\tau, ..., q_n^\tau \right]$ is determined by the consumer's preference function *f* and the period τ price vector $p^\tau = \left[p_1^\tau, p_2^\tau, ..., p_n^\tau \right]$ that the consumer faces when observing commodities or items. The consumer's preferences over the given consumption vector q are assumed to be represented by a continuous, non-decreasing, and concave utility function *f*

(CPI Manual, 2004). The main assumption here is that the consumer minimizes the cost of achieving the period τ utility level $u^\tau \equiv f\left(q^\tau\right)$ for the base period 0 and the current period t. Consequently, the consumer's cost function is defined as follows:

$$C\left(u^\tau, p^\tau\right) = min_q \left\{\sum_{i=1}^{n} p_i^\tau q_i : f\left(q\right) = u^\tau\right\}, \tau = 0, t. \tag{23}$$

The family of true cost-of-living indices (Konüs, 1924), which compares the current period with the base one, is defined as the ratio of minimum costs of achieving the same utility level $u \equiv f\left(q\right)$:

$$P_K\left(p^0, p^t, q\right) = \frac{C\left(f\left(q\right), p^t\right)}{C\left(f\left(q\right), p^0\right)}, \tag{24}$$

where $q = (q_1, q_2, \ldots, q_n)$ is a positive reference quantity vector. The homothetic preferences assumption simplifies the family of true cost-of-living indices as follows (CPI Manual, 2004):

$$P_K\left(p^0, p^t, q\right) = \frac{c\left(p^t\right)}{c\left(p^0\right)}, \tag{25}$$

where $c\left(p\right) \equiv C\left(1, p\right)$ denotes the unit cost function.

If the $c(p)$ function has a flexible functional form, i.e., it can approximate an arbitrary, twice-differentiable linearly homogeneous cost function to the second order, then the corresponding index is termed superlative (Diewert, 1976). It can be shown that the following quadratic mean of order r price index:

$$P^r\left(p^0, p^t, q^0, q^t\right) = \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^t}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^t \left(\frac{p_i^t}{p_i^0}\right)^{-r/2}}}. \tag{26}$$

is a superlative price index formula for any $r \neq 0$ (Diewert, 1976; CPI Manual, 2004).

The *implicit* quadratic mean of order price index:

$$P_{im}^r\left(p^0, p^t, q^0, q^t\right) = \frac{\sum_{i=1}^{n} p_i^t q_i^t}{Q^r\left(p^0, p^t, q^0, q^t\right) \sum_{i=1}^{n} p_i^0 q_i^0}, \tag{27}$$

where the quadratic mean of order r quantity index Q^r can be written as follows:

$$Q^r\left(p^0, p^t, q^0, q^t\right) = \frac{\sqrt[r]{\sum_{i=1}^{n} s_i^0 \left(\frac{q_i^t}{q_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i=1}^{n} s_i^t \left(\frac{q_i^t}{q_i^0}\right)^{-r/2}}}, \tag{28}$$

is also a superlative price index formula (Diewert, 1976; CPI Manual, 2004). We have: $P_{im}^{1} = P_W$, $P_{im}^{2} = P_F$, $P^2 = P_F$ and P^r ($r \to 0$) = P_T, where P_W, P_F and P_T denote the superlative Walsh, Fisher, and Törnqvist price indices, respectively.

4.2. The GEKS-IQM index class

Let us define a general GEKS-type index family as follows:

$$P_{GEKS-IQM}^{0,t}\left(r\right)=\prod_{\tau=0}^{T}\left(\frac{P_{IQM}^{\tau,t}\left(r\right)}{P_{IQM}^{\tau,0}\left(r\right)}\right)^{\frac{1}{T+1}}, \tag{29}$$

where $P_{IQM}^{\tau,t}\left(r\right)\equiv P_{im}^{r}\left(p^{\tau},p^{t},q^{\tau},q^{t}\right)$.

The indices satisfy the *time reversal test*:

$$P_{IQM}^{0,t}\left(r\right)P_{IQM}^{t,0}\left(r\right)=\frac{\sum_{i\in G_{0,t}}p_i^tq_i^t}{Q^r\left(p^0,p^t,q^0,q^t\right)\sum_{i\in G_{0,t}}p_i^0q_i^0}\frac{\sum_{i\in G_{0,t}}p_i^0q_i^0}{Q^r\left(p^t,p^0,q^t,q^0\right)\sum_{i\in G_{0,t}}p_i^tq_i^t}=$$

$$=\frac{1}{Q^r\left(p^0,p^t,q^0,q^t\right)Q^r\left(p^t,p^0,q^t,q^0\right)}=\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{q_i^t}{q_i^0}\right)^{-r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{q_i^t}{q_i^0}\right)^{r/2}}}\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{q_i^0}{q_i^t}\right)^{-r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{q_i^0}{q_i^t}\right)^{r/2}}}=$$

$$=\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{q_i^0}{q_i^t}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{q_i^t}{q_i^0}\right)^{r/2}}}\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{q_i^t}{q_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{q_i^0}{q_i^t}\right)^{r/2}}}=1.$$

It can be proved that the following theorem holds (Białek, 2025):

Theorem 3. *For each $r \neq 0$, the multilateral price index $P_{GEKS-IQM}^{0,t}\left(r\right)$ satisfies the following tests: transitivity, multi-period identity, responsiveness, continuity, positivity and normalization, homogeneity in prices, and homogeneity in quantities, as well as commensurability.*

We can obtain some known particular cases of this index class:

$$P_{GEKS-IQM}^{0,t}\left(1\right)=\prod_{\tau=0}^{T}\left(\frac{P_{W}^{\tau,t}}{P_{W}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS-W}^{0,t}, \tag{30}$$

$$P_{GEKS-IQM}^{0,t}\left(2\right)=\prod_{\tau=0}^{T}\left(\frac{P_{F}^{\tau,t}}{P_{F}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS}^{0,t}, \tag{31}$$

$$P_{GEKS-IQM}^{0,t}\left(r\to 0\right)\approx\prod_{\tau=0}^{T}\left(\frac{P_{T}^{\tau,t}}{P_{T}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{CCDI}^{0,t}. \tag{32}$$

4.3. The GEKS-QM index class

Let us define a general GEKS-type index family as follows:

$$P_{GEKS-QM}^{0,t}\left(r\right)=\prod_{\tau=0}^{T}\left(\frac{P_{QM}^{\tau,t}\left(r\right)}{P_{QM}^{\tau,0}\left(r\right)}\right)^{\frac{1}{T+1}}. \tag{33}$$

where $P_{QM}^{\tau,t}\left(r\right)\equiv P^{r}\left(p^{\tau},p^{t},q^{\tau},q^{t}\right)$.

The *QM* indices satisfy the *time reversal test*:

$$P_{QM}^{0,t}\left(r\right)P_{QM}^{t,0}\left(r\right)=\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{p_i^t}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{p_i^t}{p_i^0}\right)^{-r/2}}}\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{p_i^0}{p_i^t}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{p_i^0}{p_i^t}\right)^{-r/2}}}=$$

$$=\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{p_i^t}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{p_i^0}{p_i^t}\right)^{r/2}}}\frac{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^t\left(\frac{p_i^0}{p_i^t}\right)^{r/2}}}{\sqrt[r]{\sum_{i\in G_{0,t}}s_i^0\left(\frac{p_i^t}{p_i^0}\right)^{r/2}}}=1.$$

It can be proved that the following theorem holds (Białek, 2025):

Theorem 4. *For each $r\neq 0$ the multilateral price index $P_{GEKS-QM}^{0,t}\left(r\right)$ satisfies the following tests: transitivity, multi-period identity, responsiveness, continuity, positivity and normalization, weak price proportionality, homogeneity in prices and homogeneity in quantities, as well as commensurability.*

We may obtain some known particular cases of this index class:

$$P_{GEKS-QM}^{0,t}\left(r\rightarrow 0\right)=\prod_{\tau=0}^{T}\left(\frac{P_{T}^{\tau,t}}{P_{T}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{CCDI}^{0,t}, \tag{34}$$

$$P_{GEKS-QM}^{0,t}\left(1\right)\approx\prod_{\tau=0}^{T}\left(\frac{P_{W}^{\tau,t}}{P_{W}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS-W}^{0,t}, \tag{35}$$

$$P_{GEKS-QM}^{0,t}\left(2\right)=\prod_{\tau=0}^{T}\left(\frac{P_{F}^{\tau,t}}{P_{F}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS}^{0,t}. \tag{36}$$

4.4. The GEKS-LM index class

The Lloyd-Moulton price index (Loyd, 1975) can be written as follows:

$$P_{LM}^{\tau,t}\left(\sigma\right)=\left(\sum_{i\in G_{\tau,t}}s_i^{\tau}\left(\frac{p_i^t}{p_i^{\tau}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{37}$$

where the parameter σ denotes the elasticity of substitution. It can be shown (Loyd, 1975; Moulton, 1996) that under the assumption of cost-minimizing behavior, the index P_{LM} is exact for constant elasticity of substitution (CES) preferences. That is, for the fixed periods τ and t, it holds that

$$P_{LM}^{\tau,t}(\sigma)=\frac{c_{\sigma}\left(p^{t}\right)}{c_{\sigma}\left(p^{\tau}\right)}, \tag{38}$$

where the unit cost function $c_{\sigma}(p)$ is a CES aggregator function introduced in Arrow et al. (1961). Based on the Lloyd-Moulton index, let us define a general GEKS-type index family as follows:

$$P_{GEKS-LM}^{0,t}(\sigma)=\prod_{\tau=0}^{T}\left(\frac{P_{LM}^{\tau,t}(\sigma)}{P_{LM}^{\tau,0}(\sigma)}\right)^{\frac{1}{T+1}}, \tag{39}$$

with the following special cases:

$$P_{GEKS-LM}^{0,t}(0)=\prod_{\tau=0}^{T}\left(\frac{P_{La}^{\tau,t}}{P_{La}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS-L}^{0,t} \tag{40}$$

and

$$P_{GEKS-LM}^{0,t}(\sigma\rightarrow 1)=\prod_{\tau=0}^{T}\left(\frac{P_{GLa}^{\tau,t}}{P_{GLa}^{\tau,0}}\right)^{\frac{1}{T+1}}=P_{GEKS-GL}^{0,t}. \tag{41}$$

It can be proved that the following theorem holds (Białek, 2025):

Theorem 5. *For each $\sigma \neq 1$, the multilateral price index $P_{GEKS-LM}^{0,t}(\sigma)$ satisfies the following tests: transitivity, identity, multi-period identity, responsiveness, continuity, positivity and normalization, commensurability, strong price proportionality, as well as homogeneity in prices and homogeneity in quantities.*

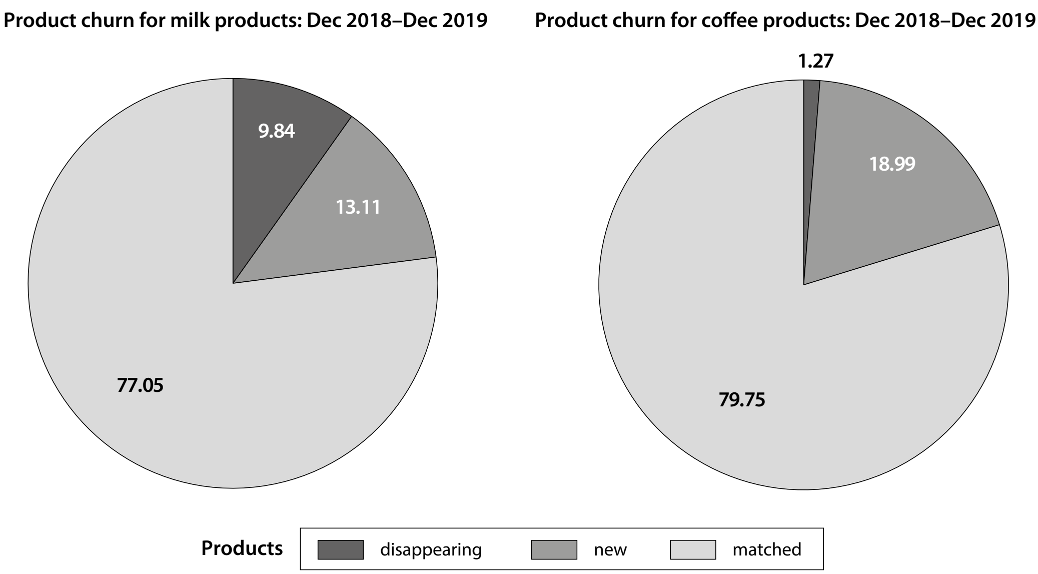
Remark 1. Please note that in this paper, we make a distinction between *strong* and *weak price proportionality tests*. To the best of the author's knowledge, this is the first distinction of this type in literature. However, this distinction is only relevant to multilateral indices, where the considerations involve many periods within a given time window. The *weak price proportionality* test assumes that price changes in all periods within the time interval are proportional to prices from the base period. This test states that in such a situation, the index comparing the current and base periods should depend only on the proportion of prices from the compared periods. In the case of a strong version of this test (i.e., the *strong proportionality test*), the same behavior of the index is expected under a less restrictive requirement because it assumes proportionality of prices in only two compared periods (prices from periods earlier than the current one do not have to be proportional to prices from the base period). It is easy to verify that a strong version of this test implies the occurrence of a weak version of the test. It can be shown that the GEKS, Geary-Khamis (Geary, 1958; Khamis, 1972), and TPD (de Haan and Krsinich, 2018) multilateral indices satisfy the *weak price proportionality test* but not the *strong price proportionality test*, while the SPQ (Diewert, 2020), GEKS-L, and GEKS-GL indices satisfy the *strong price proportionality test* (see an example in Białek (2025)).

Remark 2. In the author's opinion, there is no point in considering the *identity test, multi-period identity test* or *weak* or *strong proportionality tests* if we do not assume that the product universe is the same in the base and current periods. Of course, such an assumption does not exclude the phenomenon of product churn within the time interval under consideration.

5 EMPIRICAL ILLUSTRATION

An empirical illustration of the potential differences between the proposed indices (including representatives of general index classes) and the multilateral indices known from the literature (GEKS, CCDI, TPD, Geary-Khamis) is presented on the milk and coffee datasets. These datasets are anonymized sets of real data on coffee and milk sales in a retail chain operating in Poland. These datasets are available in the R package *PriceIndices* (Białek, 2021). The analysis was performed at the GTIN code level and covered the period Dec 2018–Dec 2019. The product rotation corresponding to these datasets and the analyzed period is presented in Fig. 1. Fig. 2 compares the indices presented in Sections 3.1, 3.2, 4.1 and 4.2 with popular multilateral indices. An analogous comparison, excluding indices from the GEKS-IQM class, is shown in Fig. 3. The next two figures (i.e., Fig. 4 and Fig. 5) consider the other two general classes of indices, i.e., the GEKS-QM and GEKS-LM.

Figure 1 Product churn in the scanner datasets considered.



Observing Figs. 2–5, one can see that, in general, the multilateral indices determined based on the *milk* and *coffee* datasets form three clusters of values. Specifically, the values of the GEKS-L, GEKS-GL, GEKS-AQI, and GEKS-AQU indices are close to each other (Cluster 1), further – the GEKS and CCDI indices approximate each other (cluster 2), and the TPD and Geary-Khamis indices have similar values (Cluster 3). In our study, it was found that values of indices from Cluster 3 are always between index values from Cluster 1 and Cluster 2. The similar values of the GEKS and CCDI index methods can be readily explained by the mutual approximation of superlative indices (Diewert, 1976), which serve as the underlying bilateral indices for these methods. However, it is particularly interesting that the indices satisfying the identity test form a distinct cluster (Cluster 1).

Figure 2 Comparison of the GEKS-L, GEKS-GL, GEKS-AQI, GEKS-AQU, and GEKS-GAQI indices with known multilateral indices.

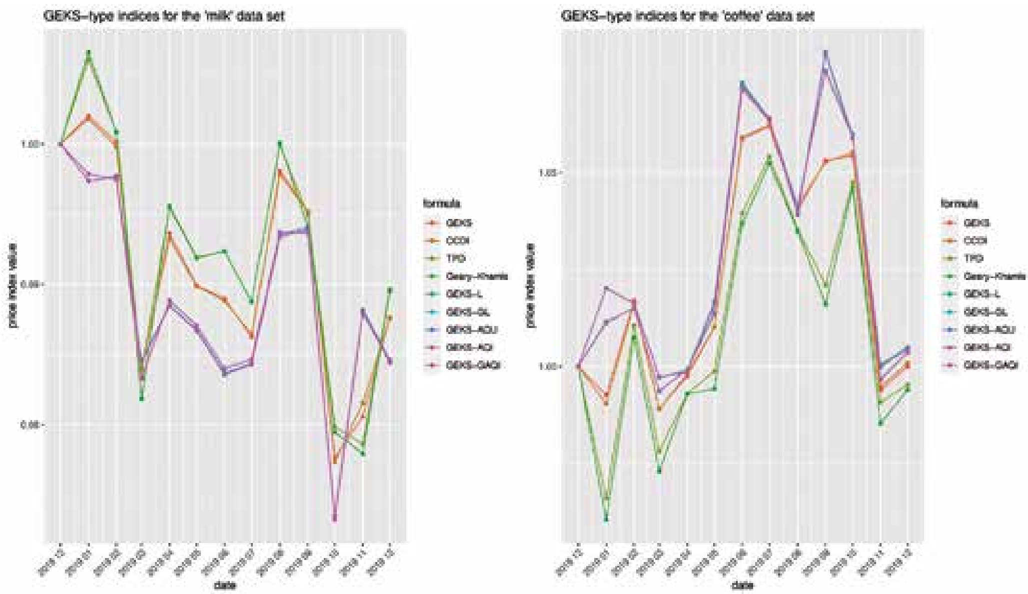


Figure 3 Comparison of the selected GEKS-IQM indices with known multilateral indices.

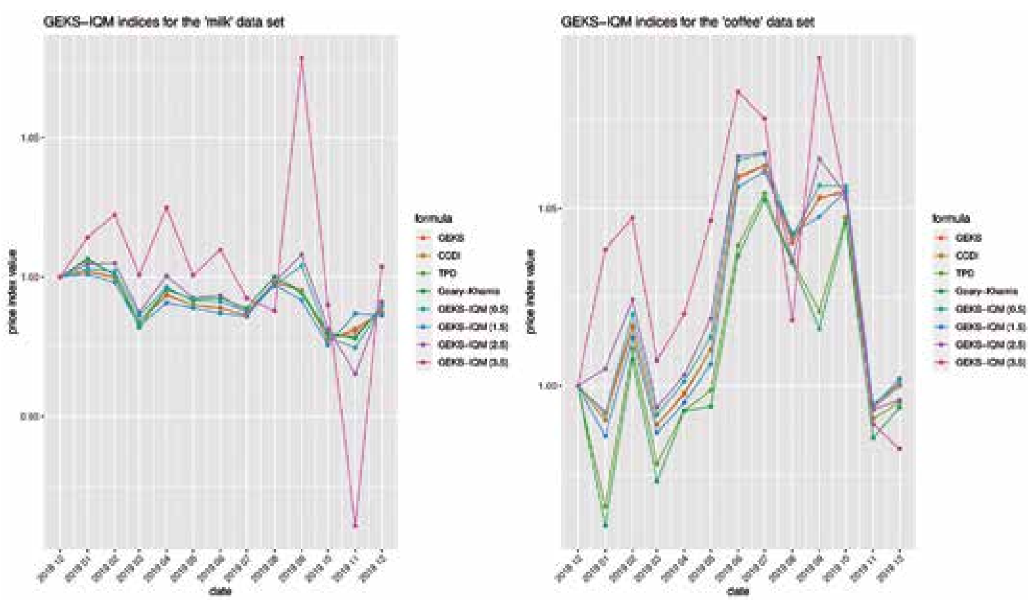


Figure 4 Comparison of the selected GEKS-QM indices with known multilateral indices.

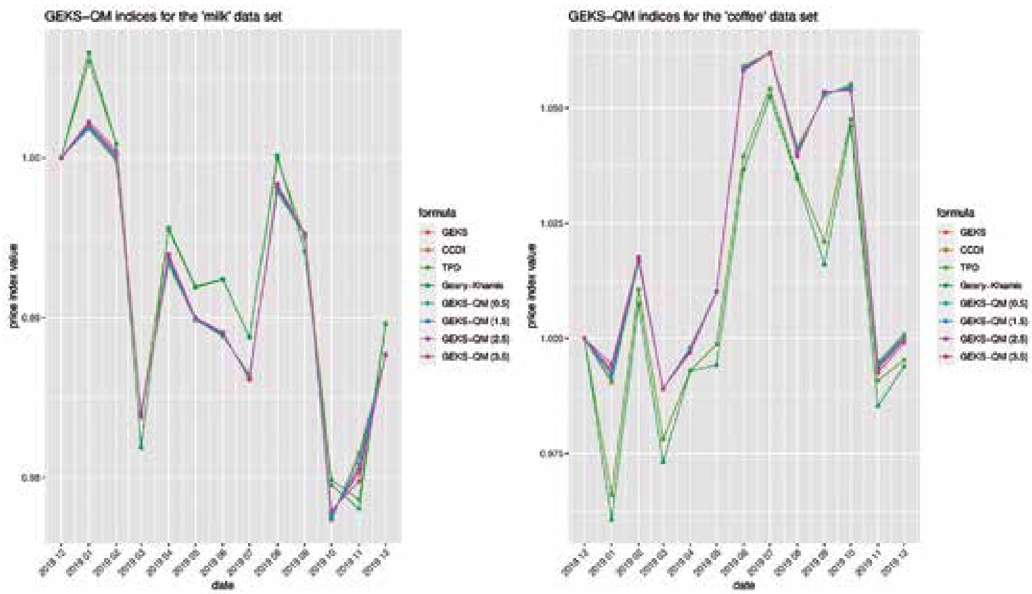
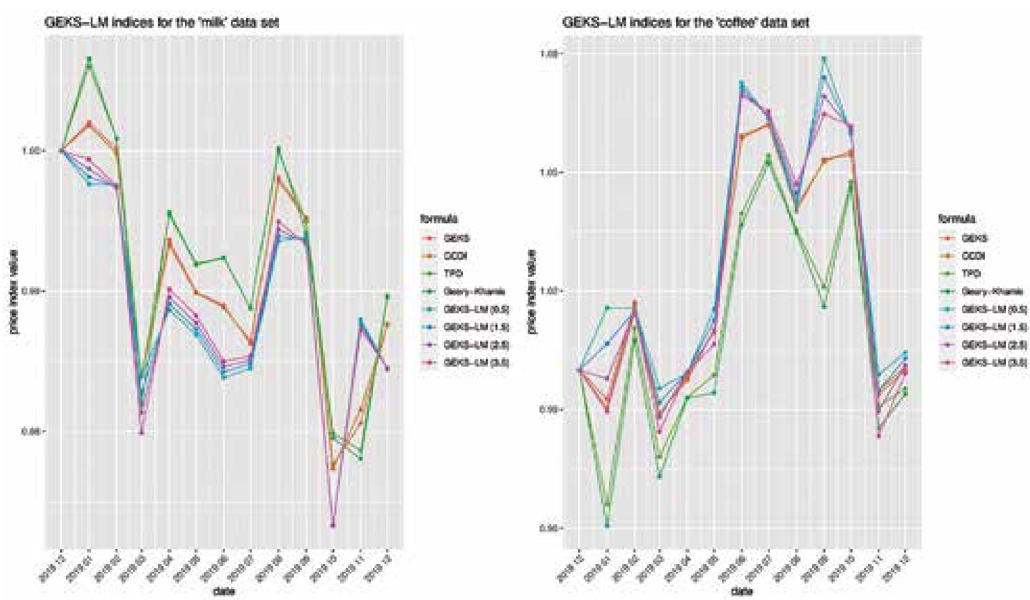


Figure 5 Comparison of the selected GEKS-LM indices with known multilateral indices.



6 SIMULATION STUDY

Our simulation study aims to investigate how price and quantity volatility affect the differences between GEKS-type indices. In the experiment, we generate price processes assuming that prices follow a geometric Brownian motion (GBM), which is commonly used for price modeling (see Ross (2014) or Hull (2018)) and whose parameters are relatively easy to estimate.

By discretizing the GBM price model, we effectively switch to a log-normal distribution, which is frequently used in price modeling, not only in finance but also in multilateral index number theory (see, for instance, Silver and Heravi (2007) or Białek (2020)). We assume that all prices at a given time (τ) are log-normally distributed, i.e.,

$$\log\left(p^{\tau}\right) \sim N\left(\mu_{\tau}, \sigma_{\tau}\right), \tag{42}$$

where $N\left(\mu_{\tau}, \sigma_{\tau}\right)$ denotes the normal distribution with mean μ_{τ} and standard deviation , σ_{τ}.

We conducted four simulation experiments, each generating artificial datasets of $N = 100$ products observed over 13 months (Dec 2023–Dec 2024). Assuming that the prices and quantities were log-normally distributed, the following values of distribution parameters were used for each month:

a) for prices $\mu^{p} = \left\{\mu_{p}^{t} : t = 0,1,2,\ldots,13\right\}$ and $\sigma^{p} = \left\{\sigma_{t}^{p} : t = 0,1,2,\ldots.,13\right\}$,
where $\mu_{p} = \{1,1.01,1.04,1.05,1.07,1.11,1.24,1.13,1.06,1.03,1.19,1.12,1.04\}$,
$\sigma^{p} = \gamma^{p}\mu^{p}\ \gamma^{p} \in \{0.05, 0.5\}$,

b) for quantities $\mu^{q} = \left\{\mu_{q}^{t} : t = 0,1,2,\ldots,13\right\}$ and $\sigma^{q} = \left\{\sigma_{t}^{q} : t = 0,1,2,\ldots.,13\right\}$, where $\mu_{t}^{1} = 6 / \mu_{t}^{p}$ and
$\sigma^{q} = \gamma^{q}\mu^{q}\ \gamma^{q} \in \{0.05, 0.5\}$

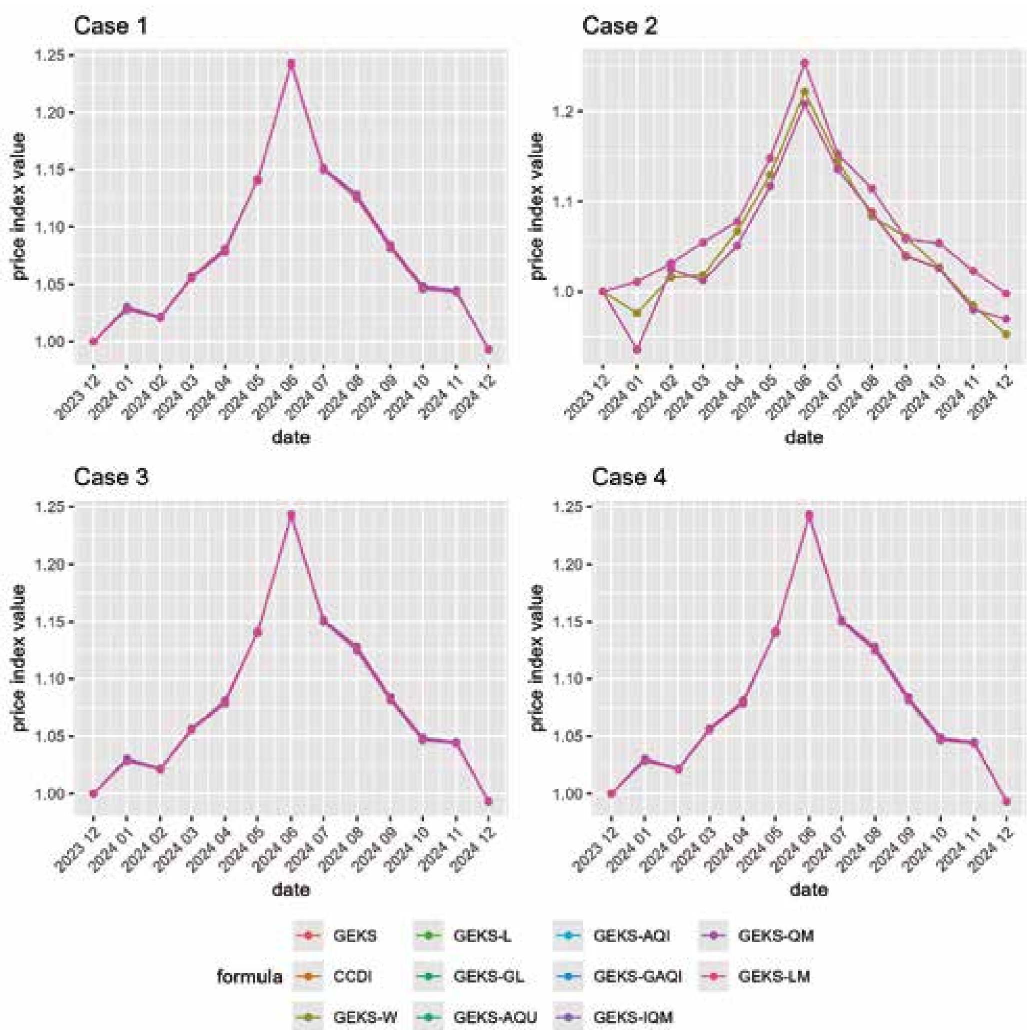
The parameters listed in (a) and (b) were selected to ensure that (1) two evident price peaks could be generated, e.g., there is a clear price increase in June 2023 and then in October 2023; (2) the price increase was accompanied by a fall in quantities (to reflect the natural negative correlation between prices and quantities); (3) it was possible to set varying levels of prices and quantities: low (Case 1: $\gamma^p = \gamma^q = 0.05$), high (Case 4: $\gamma^p = \gamma^q = 0.5$), and mixed (Case 2: $\gamma^p = 0.05$, $\gamma^q = 0.5$ and Case 3: $\gamma^p = 0.5$, $\gamma^q = 0.05$). Altogether, as already mentioned, four simulation experiments were performed.

Fig. 6 presents a comparison of the GEKS-type indices considered for Cases 1–4. The indices from the GEKS-QM and GEKS-IQM classes were determined for $r = 1.5$. The GEKS-LM index, on the other hand, was estimated for elasticity of substitution $\sigma = 0.7$. Analysis of Fig. 6 reveals that the differences between GEKS-type indices become noticeable only when there is high variation in quantities. In Case 2, where the variability of quantities is very large (and thus the levels of product consumption vary greatly), the differences between the indices considered exceeded as much as 10 percentage points (in other cases, they did not exceed 0.5 percentage points). If these indices begin to diverge, then, as in the empirical study, three clusters of GEKS-type index values emerge (see Section 5). Interestingly, even large price volatility does not generate substantial differences between the values of the GEKS-type indices (Case 3 in Fig. 6).

CONCLUSIONS

This paper expands on the results presented at the *Ottawa Group Meeting* held in Canada in 2024 (see Białek (2024) for details). The discussion of versions of the GEKS index was supplemented by a relatively new index, GEKS-GAQI, which was recently proposed by Białek and Pawelec (2025). An additional simulation study was conducted to determine the impact of price and quantity variability on potential differences between GEKS-type indices.

Figure 6 Comparison of the selected GEKS-type indices for Cases 1–4.



While the empirical illustration presented in Section 6 covers only two food groups and, thus, does not allow us to generalize conclusions, it provides some preliminary insights into the behavior of multilateral GEKS-type indices. Fig. 3 casts a bit of a shadow over the usefulness of indices from the GEKS-IQM class as they begin to lag significantly behind typical multilateral indices for the parameter . On the other hand, indices from the GEKS-QM class are stable due to the value of the parameter (Fig. 4). At the same time, differences between these indices and typical multilateral ones can be both small (no more than 0.5 p.p. as in the case of the *milk* set) and noticeable (as much as 2.5 p.p. for the *coffee* set). Fig. 5 shows that the indices from the GEKS-LM class also form a separate cluster of values and are moderately sensitive to the choice of parameter. Regarding the *coffee* data set, the indices from this class have substantially higher values than the other compared indices, with differences exceeding even 3 p.p. For the *milk* data set, indices from the GEKS-LM class generate the lowest values among the set of multilateral indices considered.

The simulation study conducted, presented in Section 6, also provides valuable practical conclusions. It was found that only large quantity volatility (and not price volatility) generates substantial differences in the values of GEKS-type indices. This conclusion differs from the observations of bilateral indices, where large price volatilities are primarily responsible for the resulting differences between the indices. In addition, the simulation study confirmed the formation of three clusters of GEKS-type index values when there are large differences between product consumption levels.

We can also draw some general conclusions about the analyzed family of multilateral indices, which may facilitate the final selection of one of the representatives of this family. From a theoretical (axiomatic) perspective, the GEKS-LM class of indices seems to be the most valuable (e.g., they satisfy the *identity test*). Important special cases of this class are the GEKS-L and GEKS-GL indices. To satisfy both the axiomatic and economic approaches simultaneously, it would be necessary to determine the elasticity of substitution parameter beforehand (e.g., by using a panel regression model that explains the behavior of quantities based on prices). This is the direction of the author's further research.

Please note that indices from the GEKS-AQU, GEKS-AQI and GEKS-GAQI classes also satisfy the *identity test*. However, a certain limitation of indices from these classes is the condition necessary to satisfy *homogeneity in prices* and *homogeneity in quantities*. This condition is a less realistic scenario, as it implies a constant set of products in the base and current periods being compared.

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