Segmented Regression Based on B-Splines with Solved Examples

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Abstract

The subject of the paper is segmented linear, quadratic, and cubic regression based on B-spline basis functions. In this article we expose the formulas for the computation of B-splines of order one, two, and three that is needed to construct linear, quadratic, and cubic regression. We list some interesting properties of these functions. For a clearer understanding we give the solutions of a couple of elementary exercises regarding these functions.

Keywords

Normalized B-spline basis functions, explicit expression, system of normal equations, Weibull plot, Bairstow's iteration method

JEL code

C63, C65

INTRODUCTION

The introduction of the paper, that constitutes its first part, is dedicated to the basic notation of B-spline functions can be found in detail in the existing literature on splines in general (see e.g. Bézier, 1972; Böhmer, 1974; Meloun, Militký, 1994; Spät, 1996). The main content of the paper lies in the aforementioned segmented regression, the theoretical background of which is given in Section 2. Here the most important part is the least squares method that leads to a system of (so-called normal) equations to compute the estimates for the parameters of the chosen regression model.

In Section 3 we describe the so-called polygonal method of value assignment of the parametric variable t (usually time) to experimentally obtained points in $\mathbb{R}^2$ or $\mathbb{R}^3$. The starting point of this method is an oriented graph with vertices given by the experimentally obtained points with the corresponding oriented edges. We associate to the graph vertices, as the value of the parameter t, the length of the polygonal trail that has its starting point in the first vertex of the graph and end point in that particular vertex. The computation of the so-called knots on the axis of the parametric variable that separate the set of experimentally obtained points into line segments (groups, sections) is automatically provided in this method.

In Section 4 we address the question of the transformation of the parametric variable into a unit-length interval, the purpose of which is to increase the numerical stability of the equations of the resulting regression curves.

In Section 5 we solve two given problems. In Example 5.2 we discuss also the notion of so-called optimal regression with respect to the coefficients of determination.

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1 B-SPLINE FUNCTIONS

There exists a large literature on B-splines, see e.g. (Bézier, 1972), however, let us fix the basic ideas about these functions that we will prefer.

By the symbol $(t)_+$ we denote the real-valued function

$$(t)_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t > 0. \end{cases}$$

A B-spline function $B_{Q,r} = B_{Q,r}(t)$ is defined for $Q \geq 1$ an integer, r an integer, and $Q + 2$ knots $T_{r-Q-1} < T_{r-Q} < \ldots < T_r$ as a normalized (Q+1)-th divided difference of the function $g(T) = [(T - t)_+)]^Q$ of real variable T. Thus, $g(T)$ is, for a given T, function of the real variable t, which we will denote as $(T - t)_+{}^Q$. Hence,

$$B_{Q,r} = (T_r - T_{r-Q-1})\,[T_{r-Q-1}, T_{r-Q}, \ldots, T_r]g. \tag{1.1}$$

The first divided difference of g is defined as

$$[T_{r-1}, T_r]g = \frac{g(T_r) - g(T_{r-1})}{T_r - T_{r-1}} = \frac{g(T_{r-1}) - g(T_r)}{T_{r-1} - T_r} = [T_r, T_{r-1}]g,$$

while the second and the third are

$$[T_{r-2}, T_{r-1}, T_r]g = \frac{[T_r, T_{r-1}]g - [T_{r-1}, T_{r-2}]g}{T_r - T_{r-2}},$$

$$[T_{r-3}, T_{r-2}, T_{r-1}, T_r]g = \frac{[T_r, T_{r-1}, T_{r-2}]g - [T_{r-1}, T_{r-2}, T_{r-3}]g}{T_r - T_{r-3}},$$

etc. Normalization of a divided difference lies in its multiplication with the corresponding denominator. More on divided differences can be found e.g. in (Schrutka, 1945). For example, for $Q = 1$ we have

$$\begin{aligned} B_{1,r} &= (T_r - T_{r-2})\cdot[T_{r-2}, T_{r-1}, T_r]g = (T_r - T_{r-2})\frac{[T_r, T_{r-1}]g - [T_{r-1}, T_{r-2}]g}{T_r - T_{r-2}} = \\ &= [T_r, T_{r-1}]g - [T_{r-1}, T_{r-2}]g = \frac{g(T_r) - g(T_{r-1})}{T_r - T_{r-1}} + \frac{g(T_{r-2}) - g(T_{r-1})}{T_{r-1} - T_{r-2}} = \\ &= \frac{(T_r - t)_+^1 - (T_{r-1} - t)_+^1}{T_r - T_{r-1}} + \frac{(T_{r-2} - t)_+^1 - (T_{r-1} - t)_+^1}{T_{r-1} - T_{r-2}}, \end{aligned}$$

that is

$$B_{1,r}(t) = \frac{T_r - t}{T_r - T_{r-1}} \qquad \text{for } T_{r-1} \leq t \leq T_r, \tag{1.2}$$

$$B_{1,r}(t) = \frac{t - T_{r-2}}{T_{r-1} - T_{r-2}} \qquad \text{for } T_{r-2} \le t \le T_{r-1}, \tag{1.3}$$

everywhere else $B_{1,r}(t)$ takes the value zero.

For the practical computation of B-spline functions we advise to use the recursive de Boor formula, see (de Boor, 1972),

$$B_{Q+1,r} = \frac{t - T_{r-Q-2}}{T_{r-1} - T_{r-Q-2}} B_{Q,r-1} + \frac{T_r - t}{T_r - T_{r-Q-1}} B_{Q,r}. \tag{1.4}$$

Example 1.1. According to (1.4), there is for $Q = 1$

$$B_{2,r}(t) = \frac{t - T_{r-3}}{T_{r-1} - T_{r-3}} B_{1,r-1} + \frac{T_r - t}{T_r - T_{r-2}} B_{1,r}. \tag{1.5}$$

We shall find the explicit expression of $B_{2,r}$ in $\langle T_{r-1}, T_r\rangle$, $\langle T_{r-2}, T_{r-1}\rangle$, $\langle T_{r-3}, T_{r-2}\rangle$.

For $t \in \langle T_{r-1}, T_r\rangle$, according to (1.2) there is

$$B_{1,r}(t) = \frac{T_r - t}{T_r - T_{r-1}},$$

while $B_{1,r-1}(t) = 0$ (because $B_{1,r-1}$ is non-zero only in (T_{r-3}, T_{r-2})). Therefore, according to (1.5)

$$B_{2,r}(t) = \frac{(T_r - t)^2}{(T_r - T_{r-1})(T_r - T_{r-2})} \qquad \text{for } T_{r-1} \le t \le T_r. \tag{1.6}$$

For $t \in \langle T_{r-2}, T_{r-1}\rangle$, according to (1.3) there is

$$B_{1,r}(t) = \frac{t - T_{r-2}}{T_{r-1} - T_{r-2}} \quad \text{and} \quad B_{1,r-1}(t) = \frac{T_{r-1} - t}{T_{r-1} - T_{r-2}}$$

(in (1.2) we replaced r by $r - 1$). According to (1.5) there is then

$$B_{2,r}(t) = \frac{(t - T_{r-3})(T_{r-1} - t)}{(T_{r-1} - T_{r-3})(T_{r-1} - T_{r-2})} + \frac{(T_r - t)(t - T_{r-2})}{(T_r - T_{r-2})(T_{r-1} - T_{r-2})} \tag{1.7}$$

for $T_{r-2} \le t \le T_{r-1}$.

For $t \in \langle T_{r-3}, T_{r-2}\rangle$, there is $B_{1,r}(t) = 0$ and

$$B_{1,r-1}(t) = \frac{(t - T_{r-3})}{T_{r-2} - T_{r-3}}$$

(in (1.3) we replaced r by $r-1$). Thus, according to (1.5),

$$B_{2,r}(t) = \frac{(t - T_{r-3})^2}{(T_{r-1} - T_{r-3})(T_{r-2} - T_{r-3})} \tag{1.8}$$

for $T_{r-3} \le t \le T_{r-2}$. Everywhere else is $B_{2,r}(t)$ zero.

Example 1.2. For $Q = 2$, according to (1.4) we have

$$B_{3,r}(t) = \frac{t - T_{r-4}}{T_{r-1} - T_{r-4}} B_{2,r-1} + \frac{T_r - t}{T_r - T_{r-3}} B_{2,r}.$$

Analogously as in Example 1.1 we find that

$$B_{3,r}(t) = \frac{(T_r - t)^3}{(T_r - T_{r-1})(T_r - T_{r-2})(T_r - T_{r-3})}, \tag{1.9}$$

for $T_{r-1} \le t \le T_r$,

$$\begin{aligned} B_{3,r}(t) = {} & \frac{(t - T_{r-4})(T_{r-1} - t)^2}{(T_{r-1} - T_{r-2})(T_{r-1} - T_{r-3})(T_{r-1} - T_{r-4})} + \\ & + \frac{(T_r - t)(t - T_{r-3})(T_{r-1} - t)}{(T_r - T_{r-3})(T_{r-1} - T_{r-2})(T_{r-1} - T_{r-3})} + \\ & + \frac{(T_r - t)^2 (t - T_{r-2})}{(T_r - T_{r-2})(T_r - T_{r-3})(T_{r-1} - T_{r-2})}, \end{aligned} \tag{1.10}$$

for $T_{r-2} \le t \le T_{r-1}$,

$$\begin{aligned} B_{3,r}(t) = {} & \frac{(t - T_{r-4})^2 (T_{r-2} - t)}{(T_{r-1} - T_{r-4})(T_{r-2} - T_{r-4})(T_{r-2} - T_{r-3})} + \\ & + \frac{(t - T_{r-3})(t - T_{r-4})(T_{r-1} - t)}{(T_{r-1} - T_{r-3})(T_{r-1} - T_{r-4})(T_{r-2} - T_{r-3})} + \\ & + \frac{(t - T_{r-3})^2 (T_r - t)}{(T_r - T_{r-3})(T_{r-1} - T_{r-3})(T_{r-2} - T_{r-3})}, \end{aligned} \tag{1.11}$$

for $T_{r-3} \le t \le T_{r-2}$, and lastly

$$B_{3,r}(t) = \frac{(t - T_{r-4})^3}{(T_{r-1} - T_{r-4})(T_{r-2} - T_{r-4})(T_{r-3} - T_{r-4})}, \tag{1.12}$$

for $T_{r-4} \le t \le T_{r-3}$. Everywhere else is $B_{3,r}(t)$ zero.

For $Q \geq 1$ whole and r whole, the functions $B_{Q,r}$ have interesting properties, see for example (Meloun, Militký, 1994):

a) They are positive only in the intervals $T_{r-Q-1} < t < T_r$ and are zero everywhere else.

b) They are normalized, i.e. for $k \geq 1$

$$\sum_{r=1}^{k+Q+1} B_{Q,r}(t) = 1 \tag{1.13}$$

in $\langle T_0, T_{k+1}\rangle$; for a complete definition of B-splines in the sum (1.13) we need to set on every side of that interval another Q so-called complementary knots

$$T_{-Q} \leq T_{-Q+1} \leq \ldots \leq T_{-1} \leq T_0, \qquad T_{k+1} \leq T_{k+2} \leq \ldots \leq T_{k+Q+1},$$

in the simplest case they merge with T_0 and T_{k+1}, respectively, on the left or right side, respectively. We call $T_1 < T_2 < \ldots < T_k$, where $T_0 < T_1$ and $T_k < T_{k+1}$, the main knots.

c) In every interval $\langle T_{s-1}, T_s\rangle$, $s = 1, 2, \ldots, k + 1$, exactly $B_{Q,s}, B_{Q,s+1}, \ldots, B_{Q,s+Q}$ are non-zero, altogether $Q + 1$ in number.

d) $B_{Q,r}$ is in $\langle T_{r-Q-1}, T_r\rangle$ polynomial spline of order Q with knots $T_{r-Q-1} < T_{r-Q} < \ldots < T_r$, i.e., in every closed interval defined by two neighbouring points $B_{Q,r}$ is a polynomial of order Q that belongs to the class $C^{Q-1}(T_{r-Q-1}, T_r)$.

We show the latter properties on the following examples.

Example 1.3. For $Q = 1$, $k = 2$, let us consider main knots $T_1 = 1$, $T_2 = 3$, complementary knots $T_{-1} = T_0 = -3{,}6 = T_3 = T_4$. According to (1.2), (1.3), we easily verify that

$$B_{1,1}(t) = \begin{cases} -\frac{1}{4}(t-1) & \text{for } -3 \leq t \leq 1, \\ 0 & \text{otherwise}, \end{cases} \tag{1.14}$$

$$B_{1,2}(t) = \begin{cases} \frac{1}{4}(t+3) & \text{for } -3 \leq t \leq 1, \\ -\frac{1}{2}(t-3) & \text{for } 1 \leq t \leq 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{1,3}(t) = \begin{cases} \frac{1}{2}(t-1) & \text{for } 1 \leq t \leq 3, \\ -\frac{1}{3}(t-6) & \text{for } 3 \leq t \leq 6, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{1,4}(t) = \begin{cases} \frac{1}{3}(t-3) & \text{for } 3 \leq t \leq 6, \\ 0 & \text{otherwise}. \end{cases}$$

For example, $B_{1,3}$ is positive only in $(T_{3-1-1}, T_3) = (T_1, T_3) = (1{,}6)$, everywhere else is zero; see a). For $s = 2$, in $\langle T_{s-1}, T_s\rangle = \langle T_1, T_2\rangle = \langle 1{,}3\rangle$ only $B_{1,2}$ and $B_{1,3}$ non zero; see c).

For example, for $t_0 = \frac{7}{2} \in \langle T_2, T_3\rangle$, where $\langle T_2, T_3\rangle = \langle T_{s-1}, T_s\rangle$ for $s = 3$, only $B_{1,3}$ and $B_{1,4}$ are non-zero, while

$$B_{1,s}\left(\frac{7}{2}\right) = B_{1,3}\left(\frac{7}{2}\right) = -\frac{1}{3}\left(\frac{7}{2} - 6\right) = \frac{5}{6},$$

$$B_{1,s+1}\left(\frac{7}{2}\right) = B_{1,4}\left(\frac{7}{2}\right) = \frac{1}{3}\left(\frac{7}{2} - 3\right) = \frac{1}{6}.$$

Hence,

$$\sum_{r=1}^{k+Q+1} B_{Q,r}(t_0) = \sum_{r=1}^{4} B_{1,r}\left(\frac{7}{2}\right) = B_{1,3}\left(\frac{7}{2}\right) + B_{1,4}\left(\frac{7}{2}\right) = \frac{5}{6} + \frac{1}{6} = 1;$$

see b). For $r = 3$, the B-spline $B_{1,3}$ is in the interval $\langle T_{r-Q-1}, T_r\rangle = \langle T_1, T_3\rangle = \langle 1,6\rangle$ of class C^{Q-1} (T_1, T_3) $= C^0$ (T_1, T_3), i.e., continuous in this interval. For example, for $t_0 = 3 \in \langle 1,6\rangle$ we have $B_{1,3}\,(3-) = 1 = B_{1,3}\,(3+)$; see d).

Example 1.4. For $Q = 2$, $k = 3$ let us consider main knots $T_1 = 3$, $T_2 = 6$, $T_3 = 9$ together with complementary knots $T_{-2} = T_{-1} = T_0 = 0$ and $12 = T_4 = T_5 = T_6$. According to (1.6), (1.7), (1.8), we easily find that

$$B_{2,1}(t) = \begin{cases} \frac{1}{9}(t^2 - 6t + 9) & \text{for } 0 \le t \le 3, \\ 0 & \text{otherwise,} \end{cases} \tag{1.15}$$

$$B_{2,2}(t) = \begin{cases} -\frac{1}{18}(3t^2 - 12t) & \text{for } 0 \le t \le 3, \\ \frac{1}{18}(t^2 - 12t + 36) & \text{for } 3 \le t \le 6, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{2,3}(t) = \begin{cases} \frac{1}{18}t^2 & \text{for } 0 \le t \le 3, \\ -\frac{1}{18}(2t^2 - 18t + 27) & \text{for } 3 \le t \le 6, \\ \frac{1}{18}(t^2 - 18t + 81) & \text{for } 6 \le t \le 9, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{2,4}(t) = \begin{cases} \frac{1}{18}(t^2 - 6t + 9) & \text{for } 3 \le t \le 6, \\ -\frac{1}{18}(2t^2 - 30t + 99) & \text{for } 6 \le t \le 9, \\ \frac{1}{18}(t^2 - 24t + 144) & \text{for } 9 \le t \le 12, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{2,5}(t) = \begin{cases} \frac{1}{18}(t^2 - 12t + 36) & \text{for } 6 \le t \le 9, \\ -\frac{1}{18}(3t^2 - 60t + 288) & \text{for } 9 \le t \le 12, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{2,6}(t) = \begin{cases} \frac{1}{9}(t^2 - 18t + 81) & \text{for } 9 \le t \le 12, \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $t_0 = \frac{13}{3} \in \langle T_1, T_2\rangle$, where $\langle T_1, T_2\rangle = \langle T_{s-1}, T_s\rangle$ for $s = 2$, only $B_{2,2}$, $B_{2,3}$ and $B_{2,4}$ are non-zero in this interval, and

$$B_{2,2}\left(\frac{13}{3}\right) = \left(\frac{25}{162}\right), \quad B_{2,3}\left(\frac{13}{3}\right) = \frac{121}{162}, \quad B_{2,4}\left(\frac{13}{3}\right) = \frac{16}{162}.$$

Thus,

$$\sum_{r=1}^{k+Q+1} B_{Q,r}(t_0) = \sum_{r=1}^{6} B_{2,r}\left(\frac{13}{3}\right) = B_{2,2}\left(\frac{13}{3}\right) + B_{2,3}\left(\frac{13}{3}\right) + B_{2,4}\left(\frac{13}{3}\right) = \frac{25}{162} + \frac{121}{162} + \frac{16}{162} = 1;$$

see b).

For $r = 4$, in the interval $\langle T_{r-Q-1}, T_r\rangle = \langle T_1, T_4\rangle = \langle 3, 12\rangle$ the function $B_{2,4}$ belongs to $C^1(T_1, T_4)$, i.e., it has a continuous derivative in this interval. E.g., at $t_0 = 9 \in \langle 3, 12\rangle$ the left derivative of this function is $-\frac{1}{3}$, while its right derivative is also $-\frac{1}{3}$; see d).

Example 1.5. For $Q = 3$, $k = 4$, let us consider main knots $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, $T_4 = 4$ together with complementary nodes $T_{-3} = T_{-2} = T_{-1} = T_0 = -1$, $6 = T_5 = T_6 = T_7 = T_8$. As before, we get

$$B_{3,1}(t) = \begin{cases} -\frac{1}{8}(t^3 - 3t^2 + 3t - 1) & \text{for } -1 \le t \le 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.16}$$

$$B_{3,2}(t) = \begin{cases} \frac{1}{72}(19t^3 - 33t^2 - 15t) + 37) & \text{for } -1 \le t \le 1, \\ -\frac{1}{9}(t^3 - 6t + 12t - 8) & \text{for } 1 \le t \le 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{3,3}(t) = \begin{cases} -\frac{1}{72}(13t^3 - 3t^2 + 33t - 23) & \text{for } -1 \le t \le 1, \\ \frac{1}{72}(23t^3 - 111t^2 + 141t - 13) & \text{for } 1 \le t \le 2, \\ -\frac{1}{8}(t^3 - 9t^2 + 27t - 27) & \text{for } 2 \le t \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{3,4}(t) = \begin{cases} \frac{1}{24}(t^3 - 3t^2 + 3t + 1) & \text{for } -1 \le t \le 1, \\ -\frac{1}{72}(27t^3 - 99t^2 + 81t - 33) & \text{for } 1 \le t \le 2, \\ \frac{1}{24}(11t^3 - 87t^2 + 213t - 149) & \text{for } 2 \le t \le 3, \\ -\frac{1}{6}(t^3 - 12t^2 + 48t - 64) & \text{for } 3 \le t \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

For capacity reasons, for $B_{3,5}, \ldots, B_{3,8}$ we do not provide here their expression.

2 SEGMENTED LINEAR, QUADRATIC, AND CUBIC REGRESSION

Let $n \ge 2$ be an integer. In the Euclidean space $\mathbb{R}^m$ (for integer $m > 1$) let us consider n points $P_i = (x_1^{(i)}, \ldots, x_m^{(i)}) = x_j^{(i)}$, $i = 1, \ldots, n$ (to save space, here and in what follows j will represent the numbers $1, 2, \ldots, m$) where at least two are different, obtained during a specific experiment.

Besides these points, $x_j^{(i)}$, $j = 1, \ldots, m$, are assumed to be real random variables, consider further knots $T_1 < T_2 < \ldots < T_k$, $k \ge 1$ an integer, and $T_0 < T_1$, $T_{k+1} > T_k$ complementary knots. As in Section 1, we call $T_1 < T_2 < \ldots < T_k$ main knots.

In the interval $\langle T_{l-1}, T_l\rangle$ for $l = 1, \ldots, k + 1$ where the variable t changes, let us consider and increasing sequence $t_{l1} < t_{l2} < \ldots < t_{l,n(l)}$, $n(l) \ge 1$ an integer, while each its member corresponds to one point $x_j^{(w)}$, $w = 1, \ldots, n(l)$. It holds that $n = \sum_{l=1}^{k+1} n(l)$. The knots form the interval boundaries, in the union of which we will consider depending on the number $Q = 1, 2, 3$ a real function of variable t

$$g_j(t) = \sum_{r=1}^{k+Q+1} \gamma_j^{(r)} B_{Q,r}(t), \tag{2.1}$$

for real parameters $\gamma_j^{(r)}$ and $B_{Q,r}$ B-splines, $r = 1, \dots, k + Q + 1$; see Section 1. For $Q = 1$ we say that (2.1) is a linear spline in the form of B-splines, for $Q = 2$ quadratic, and for $Q = 3$ cubic spline in the aforementioned form.

We will assume that the model of the monitored process is additive, that is, for all values of j, l, w under consideration, it holds that

$$x_j^{(lw)} = g_j(t_{lw}) + \varepsilon_j^{(lw)},$$

where $\varepsilon_j^{(lw)}$ are independent and identically distributed random variables with constant variance. So the estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$ can be obtained by the least squares method:

$$U_j = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} [x_j^{(lw)} - g_j(t_{lw})]^2 = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} \left[x_j^{(lw)} - \sum_{r=1}^{k+Q+1} \gamma_j^{(r)} B_{Q,r}(t_{lw}) \right]^2. \tag{2.2}$$

Differentiating (2.2) partially with respect to the parameters, for $1 \le p \le k + Q + 1$, we get

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = -2 \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} [x_j^{(lw)} - g_j(t_{lw})] \cdot B_{Q,p}(t_{lw}). \tag{2.3}$$

It is known from mathematical analysis that the necessary condition for U_j, as a function of the parameters, $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$, to attain its minimum is given by the system of equations

$$\frac{\partial U_j}{\partial \gamma_j^{(p)}} = 0 \quad \text{for } p = 1, \dots, k + Q + 1.$$

This yields through nullification of (2.3) a system of $k + Q + 1$ linear equations for the estimates $c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(k+Q+1)}$:

$$\boldsymbol{M}\boldsymbol{c}_j = \boldsymbol{Z}_j, \tag{2.4}$$

where $M = (m_{pq})_{1\le p,q\le k+Q+1}$ is a $(k+Q+1) \times (k+Q+1)$ matrix, $Z_j = (z_{pj})_{1\le p\le k+Q+1}$ and $c_j = (c_j^{(p)})_{1\le p\le k+Q+1}$ are p-dimensional vectors.

The structure of $\boldsymbol{M}$ and $\boldsymbol{Z}_j$ depends on the type of $g_j(t)$, see (2.1). If we put, for brevity, $N_r = B_{Q,r}$ then after nullification of (2.3) we arrive to the expression of the components of $\boldsymbol{M}$ and $\boldsymbol{Z}_j$:

$$m_{pq} = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} N_p(t_{lw}) \cdot N_q(t_{lw}), \tag{2.5}$$

$$z_{pj} = \sum_{l=1}^{k+1} \sum_{w=1}^{n(l)} x_j^{(lw)} N_q(t_{lw}). \tag{2.6}$$

From (2.5) it follows that M is symmetric. For $Q = 1$ it is always tridiagonal, five-diagonal for $Q = 2$, and seven-diagonal for $Q = 3$. Such systems of equations can be solved by a recursive procedure which is stable in the sense of error accumulation, see (Makarov, Chlobystov, 1983); the existence of a main diagonal for $\boldsymbol{M}$ means according to definition that

$$\min_{p} \left\{ |m_{pp}| - \sum_{q \neq p} |m_{pq}| \right\} > 0.$$

After solving the system (2.4), we acquire the sought estimates $c_j^{(1)}, c_j^{(2)}, \dots , c_j^{(k+Q+1)}$ of parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \dots , \gamma_j^{(k+Q+1)}$ in the linear combination $g_j(t)$, $t \in \langle T_0, T_{k+1} \rangle$, see (2.1). The corresponding regression spline to these estimates (linear for $Q = 1$, quadratic for $Q = 2$, and cubic for $Q = 3$), for $t \in \langle T_0, T_{k+1} \rangle$, admits the equations

$$x_j = G_j(t) = \sum_{r=1}^{k+Q+1} c_j^{(r)} B_{Q,r}(t). \tag{2.7}$$

To summarize (for $j = 1, \dots , m$), these equations represent the parametric expression of a curve in $\mathbb{R}^m = (0; x_1, x_2, \dots , x_m)$, which is the output of the regression model of the monitored process; we will call it, in short, a regression curve (linear for $Q = 1$, quadratic for $Q = 2$, and cubic for $Q = 3$).

Due to the special structure of the matrix of the (normalized) system of equations (2.4), that is three-diagonal for $Q = 1$, five-diagonal for $Q = 2$, seven-diagonal for $Q = 3$, the author of the article decided for segmented regression based on B-spline basis functions; such a matrix, the elements of which are all zero except for the given diagonals, enables for an easier and faster computation of the sought solution.

Example 2.1. Let us assume that there were values of the following two parameters: temperature and air pressure detected during 24 hours, every two hours beginning at 6 am, at a given place. Table 1 states the results of this measurement.

Table 1 Fictitious temperature and pressure measurement over a 24h period that could represent a real-world experiment

Time [*h*]		Temperature	Pressure
real	**fictitious**	**[°C]**	**[hPa]**
6:00	0	15	800
8:00	1	16	850
10:00	2	17	900
12:00	3	22	1 000
14:00	4	28	1 050
16:00	5	26	1 020
18:00	6	20	950
20:00	7	19	900
22:00	8	18	890
24:00	9	16	840
2:00	10	15	820
4:00	11	13	810

Source: Own construction

In $\mathbb{R}^2$ (hence, $j = 1, 2$) we have 12 experimentally obtained points, split in four groups (hence, $k = 3$) by three points (we may call them morning, noon, evening and night group):

$$x_j^{(11)} = (15, 800), \quad x_j^{(12)} = (16,850), \quad x_j^{(13)} = (17,900),$$

$$x_j^{(21)} = (22,1000), \quad x_j^{(22)} = (28,1050), \quad x_j^{(23)} = (26,1020),$$

$$x_j^{(31)} = (20,950), \quad x_j^{(32)} = (19,900), \quad x_j^{(33)} = (18,890),$$

$$x_j^{(41)} = (16,840), \quad x_j^{(42)} = (15,820), \quad x_j^{(43)} = (13,810),$$

to which we assign the following (increasing) time values

$$t_{11} = 0, \quad t_{12} = 1, \quad t_{13} = 2, \quad \text{thus} \quad n(1) = 3,$$

$$t_{21} = 3, \quad t_{22} = 4, \quad t_{23} = 5, \quad \text{thus} \quad n(2) = 3,$$

$$t_{31} = 6, \quad t_{32} = 7, \quad t_{33} = 8, \quad \text{thus} \quad n(3) = 3,$$

$$t_{41} = 9, \quad t_{42} = 10, \quad t_{43} = 11, \quad \text{thus} \quad n(4) = 3,$$

see Table 1. It is the case of three main knots, their values can be $T_1 = 3$, $T_2 = 6$, $T_3 = 9$, together, for example, with additional time moments $T_0 = 0$ and $T_4 = 12$.

For example, for $Q = 2$, the matrix $\boldsymbol{M}$ of the system (2.4) is a 6×6 matrix, note that $k + Q + 1 = 3 + 2 + 1 = 6$. To save space, we neither give its full expression, nor for $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$. This computationally intensive work was conducted by the computer program TRIO, that the author of this article created for the purposes of segmented regression based on B-splines.

Nevertheless, for demonstration purposes, let us compute the element m_{56} of $\boldsymbol{M}$ in accord with (2.5). There will be

$$\begin{aligned} m_{56} &= \sum_{l=1}^{4} \sum_{w=1}^{3} N_5(t_{lw}) \cdot N_6(t_{lw}) = \sum_{s=0}^{11} N_5(s) \cdot N_6(s) \\ &= N_5(9) \cdot N_6(9) + N_5(10) \cdot N_6(10) + N_5(11) \cdot N_6(11), \end{aligned} \tag{2.8}$$

as all the other terms of the sum vanish, see (1.15). According to (1.15), there are

$$N_5(9) \cdot N_6(9) = \frac{1}{2} \cdot 0, \qquad N_5(10) \cdot N_6(10) = \frac{2}{3} \cdot \frac{1}{9}, \qquad N_5(11) \cdot N_6(11) = \frac{1}{2} \cdot \frac{4}{9},$$

hence

$$m_{56} = 0 + \frac{2}{27} + \frac{2}{9} = \frac{2+6}{27} = \frac{8}{27} = 0.2963E + 00.$$

Let us also compute the component z_{62} of the vector $\boldsymbol{Z}_2$. According to (2.6), there will be

$$z_{62} = \sum_{l=1}^{4} \sum_{w=1}^{3} x_2^{(lw)} N_6\,(t_{lw}) = x_2^{(41)} N_6\,(9) + x_2^{(42)} N_6\,(10) + x_2^{(43)} N_6\,(11), \tag{2.9}$$

as all the other terms of the sum are zero. Thus,

$$z_{62} = 840 \cdot 0 + 820 \cdot \frac{1}{9} + 810 \cdot \frac{4}{9} = \frac{820 + 3240}{9} = \frac{4060}{9} = 4.5111E + 02.$$

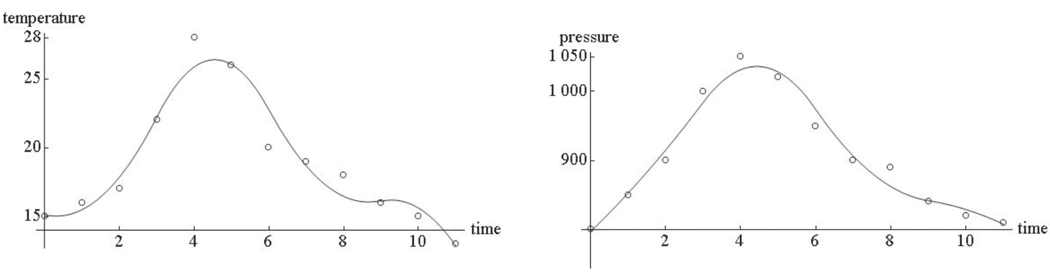
The parametric equations of the regression curve, compare with (2.7), which were obtained with the aid of the computer program TRIO, are the following:

$$x_1 = G_1\,(t) = \begin{cases} 15.1238 - 0.6221t + 09999t^2 & \text{for } 0 \le t < 3, \\ -9.5371 + 15.8185t - 1.7402t^2 & \text{for } 3 \le t < 6, \\ 87.3552 - 16.4789t + 0.9521t^2 & \text{for } 6 \le t < 9, \\ -81.5195 + 21.0488t - 1.1336t^2 & \text{for } 9 \le t \le 12, \end{cases}$$

and

$$x_2 = G_2\,(t) = \begin{cases} 797.0927 + 52.2798t + 3.3781t^2 & \text{for } 0 \le t < 3, \\ 538.6311 + 224.5875t - 25.3398t^2 & \text{for } 3 \le t < 6, \\ 1876.4612 - 221.3558t + 11.8221t^2 & \text{for } 6 \le t < 9, \\ 574.7062 + 67.9230t - 4.2489t^2 & \text{for } 9 \le t \le 12. \end{cases}$$

Figure 1 B-spline approximation for the temperature and pressure



Source: Own computation

For example, to the value $t = t_{23} = 5$ corresponds on the regression curve (in the plane $(0;\, x_1\, x_2)$) the point (26.0504,1028.0736), which lies "near" the point (26,1020) of the experiment. Or to $t = 8.5$ corresponds on the regression curve the point (16.0088, 849.0836). We can infer that one hour before midnight the air temperature was approximately 16°C and the air pressure was approximately 850 hPa.

3 THE POLYGONAL METHOD

By polygonal method we shall call in short the following procedure of assigning values of t, our "operating" variable (usually time), to the experimentally obtained points.

In $\mathbb{R}^2$ let us consider the planar connected oriented graph $\vec{G} = [A, \vec{B}]$ with the set of vertices $A = \{1, 2, \dots, n\}$, $n \geq 2$, and $\vec{B} = \{(1, 2),(2, 3), \dots, (n - 1, n)\}$, the set of (oriented) edges. We could imagine that the planar polygonal trail obtained in this way, with starting point in 1 and end point in n, is an idealized route of a car moving by constant speed, which started from point 1 and ended the journey at n. Each vertex of the graph $\vec{G}$ can be thought of as trial points, the position of which in the map we find by measuring its distance (for example in km) from the left and bottom edges of the map. We divide the vertices of the graph into $k + 1$ groups, for $k \geq 1$, by $n(l) \geq 1$ points $x_j^{(lw)}$ ($l = 1, \dots, k + 1$; $w = 1, \dots, n(l)$; $j = 1, 2$) in such a way that

$$n = \sum_{l=1}^{k+1} n(l)$$

(this division of the vertices might be caused, e.g., by the difficulty of the corresponding road terrain), and we assign to them an (increasing) sequence of values t_{lw} (in km), where t_{lw} indicates the length of the accomplished route from the start at 1 to the place at $x_j^{(lw)}$, that can be thought of as a resting place during the drive.

We include the values $t_{l1} < t_{l2} < \dots < t_{l,n(l)}$, for $l = 1, \dots, k + 1$, into intervals $\langle T_{l-1}, T_l)$. We further demand that $T_0, T_1, \dots, T_k, T_{k+1}$ is an increasing sequence such that $T_{l-1} \leq t_{l1}$, for $l = 1, \dots, k + 1$ (we shall call $T_1, \dots, T_k$ main knots, while $T_0 < T_1$, $T_{k+1} > T_k$ complementary knots for the observed drive; compare with Section 2).

It is meaningful to set $T_0 = 0$, further, from $T_l \leq t_{l+1,1}$ it follows after substituting for $T_l = \lfloor t_{l,n(l)} \rfloor + p_l \leq t_{l+1,1}$ that $p_l \leq t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor$ (where $\lfloor x \rfloor$ denotes the integer part of the real number x). Let

$$P = \min_{l = 1,\dots,k} \{t_{l+1,1} - \lfloor t_{l,n(l)} \rfloor\} \tag{3.1}$$

and $p = \lfloor P \rfloor$. If $p \geq 1$, then we set $p_l = p$, for $l = 1, \dots, k + 1$; we shall return to the case when $p = 0$.

Putting aside the drive route, we may say that the polygonal method presents a certain automatization in the assignment of operating-variable values to experimental points, divided by a given procedure into groups, that includes the computation of knots defining the range of assigned values to groups (in the aforementioned car drive example the operating variable is the length of the passed track). This polygonal method is implemented in the program TRIO and is capable of solving segmented regression problems in $\mathbb{R}^2$ and $\mathbb{R}^3$, as well.

Example 3.1. In $\mathbb{R}^3$ let us consider the following points $x_j^{(lw)}$ ($l = 1, 2, 3$; $w = 1, \dots, n(l)$, where $n(1) = 2$, $n(2) = 3$, $n(3) = 2$) divided into three groups:

$$x_j^{(11)} = (1, 1, 1), \qquad x_j^{(12)} = (1.1, 1.2, 1.3),$$

$$x_j^{(21)} = (1.5, 1, 1.4), \qquad x_j^{(22)} = (2, 3, 4), \qquad x_j^{(23)} = (3, 3, 5),$$

$$x_j^{(31)} = (3.1, 3.2, 5.05), \qquad x_j^{(32)} = (4, 4, 6).$$

Through the polygonal method we assign (increasing) operating-variable values to them (that can be, for example, time):

$t_{11} = 0.0000,$ $t_{12} = 0.3742,$

$t_{21} = 0.8325,$ $t_{22} = 4.1506,$ $t_{23} = 5.5648,$

$t_{31} = 5.7939,$ $t_{32} = 7.3277,$

According to (3.1), there is $P = \min\{0.8325, 0.7939\} = 0.7939$, hence $p = \lfloor P \rfloor = 0$.

We will proceed further as follows. We replace the points $x_j^{(lw)}$ considered above by $\tilde{x}_j^{(lw)} = L \cdot x_j^{(lw)}$, where $L > 1$ is a sufficiently large number, and through the polygonal method we assign to them (increasing) operating-variable values $\tilde{t}_{lw} = L \cdot t_{lw}$. For example, for $L = 10$ we obtain now $P = \min\{5.3242, 2.9390\} = 2.9390$, hence $p = \lfloor P \rfloor = 2$. We get the following knots that will we applied to the desired segmented regression (for which the program TRIO is ready): $\tilde{T}_0 = 0$, $\tilde{T}_1 = 5$, $\tilde{T}_2 = 57$, $\tilde{T}_3 = 75$, to which in the initial situation correspond the knots $T_0 = 0$, $T_l = \frac{\tilde{T}_l}{10}$, $l = 1, 2, 3$, i.e. $T_0 = 0$, $T_l = 0.5$, $T_2 = 5.7$, $T_3 = 7.5$.

It is worth to note one more remark. It might happen that the computed knot T_{k+1} will be too far to the right from the length of the entire polygonal trail processed by the computer. The program TRIO enables in this case its reduction to the demanded size.

4 THE TRANSFORMATION OF THE PARAMETRIC (OPERATING) VARIABLE

The elements of $\boldsymbol{M}$ and $\boldsymbol{Z}_j$ in the system of equations (2.4) are structured by the fact that we are working with B-splines. For the improvement of numerical stability of the parametric equations of the regression curve (compare with (2.7)) that is the result of the used regression model, it is recommended in the literature to transform the respective parameter into a unit-length interval (if the length of interval for the initial parameter is much larger than 1; see for example (Meloun, Militký, 1994)). Let us remind that, vaguely speaking, numerical stability of a computational process means "reasonable" or "unreasonable" loss of decimals during the computation.

We transform $t \in \langle T_0, T_{k+1} \rangle$ into

$$t' = M + \frac{M - N}{T_{k+1} - T_0}(t - T_{k+1}) = M + K(t - T_{k+1}) = f(t), \tag{4.1}$$

where $0 \le N < M$ are real numbers and $K = \frac{M-N}{T_{k+1} - T_0} > 0$.We can easily see that for any two values $t_1 < t_2$ from this interval it holds that

$$f(t_2) - f(t_1) = K(t_2 - t_1). \tag{4.2}$$

For $l = 1, \dots, k + 1$, the interval $\langle T_{l-1}, T_l \rangle$, where t is changing, transforms onto the interval $\langle T'_{l-1}, T'_l \rangle$, where the variable to change will be t'.

In general, for $Q = 1, 2, 3$ and integer $k \ge 1$, for B-splines $N'_{l+s-1}(t') = B_{Q,l+s-1}(t')$, where $s = 1,\dots,Q + 1$, which are non-zero in $\langle T'_{l-1}, T'_l \rangle$, it holds that

$$N'_{l+s-1}(t') = N_{l+s-1}(t) = N_{l+s-1}(f^{-1}(t')). \tag{4.3}$$

Indeed, for example, for $Q = 3$, $k = 2$, $s = 4$, $l = 3$ we get, according to (1.12) and (4.2), that

$$N_6'(t') = B_{3,6}'(t') = \frac{(t'-T_2')^3}{(T_5'-T_2')(T_4'-T_2')(T_3'-T_2')}$$

$$= \frac{[f(t)-f(T_2)]^3}{[f(T_5)-f(T_2)][f(T_4)-f(T_2)][f(T_3)-f(T_2)]} =$$

$$= \frac{K^3(t-T_2)^3}{K^3(T_5-T_2)(T_4-T_2)(T_3-T_2)} = N_6(t) = N_6(f^{-1}(t')),$$

for $T'_2 \leq t' \leq T'_3$, that is, for $T_2 \leq t = f^{-1}(t') \leq T_3$. According to (4.3), the transformation $t' = f(t)$ of the interval $\langle T_1, T_{k+1}\rangle$ onto $\langle N, M\rangle$ does not change the system of normal equations (2.4) (for $j = 1, \ldots, m$, $Q = 1, 2, 3$ and integer $k \geq 1$), it provides, therefore, the same estimates $c_j^{(1)}, c_j^{(1)}, \ldots, c_j^{(k+Q+1)}$ of the parameters $\gamma_j^{(1)}, \gamma_j^{(2)}, \ldots, \gamma_j^{(k+Q+1)}$ in the linear combination of B-splines

$$g_j'(t') = \sum_{r=1}^{k+Q+1} \gamma_j^{(r)} B_{Q,r}'(t'),$$

as in the untransformed case (2.1). The regression spline corresponding to these estimates (linear for $Q = 1$, quadratic for $Q = 2$, cubic for $Q = 3$) admits, for $t' \in \langle T'_0 = N, T'_{k+1} = M\rangle$ the equation

$$x_j = G_j'(t') = \sum_{r=1}^{k+Q+1} c_j^{(r)} B_{Q,r}(f^{-1}(t')). \tag{4.4}$$

To summarize, the equations (4.4) represent, for $j = 1, \ldots, m$, the parametric expression of the same regression curve (linear for $Q = 1$, quadratic for $Q = 2$, and cubic for $Q = 3$) as equations (2.7).

5 TWO EXAMPLES

Example 5.1. Let us provide, using a Weibull plot, the failure analysis of lining pads of front disc brakes of cars based on real values observed for cars in Federal Republic of Germany (see (VDA3, 1995)). The goal is to determine the characteristic lifetime defined as the lifetime until which 63.2121% of monitored units is broken ($63.2121 = (1 - e^{-1}) \cdot 100$).

The starting point is Table 2. For the mean order number, median order there are in (VDA3, 1995) available the corresponding formulas. We display the points (t_q km · 1000, H_q%), for $q = 1, \ldots, 30$, in a Weibull plot, divided e.g. into three groups of ten (in accord with previous notations, $k = 2$, see Section 2):

$$x_j^{(11)} = (8.8, 1.74), \quad x_j^{(12)} = (10.3, 6.02), \quad \ldots, \quad x_j^{(1,10)} = (16.4, 31.54),$$

$$x_j^{(21)} = (17.7, 33.79), \quad x_j^{(22)} = (19.3, 36.13), \quad \ldots, \quad x_j^{(2,10)} = (29.9, 59.09),$$

$$x_j^{(31)} = (30.4, 62.03), \quad x_j^{(32)} = (32.1, 64.98), \quad \ldots, \quad x_j^{(3,10)} = (55.7, 97.48),$$

And we assign to them values of an (operating) variable t through the polygonal method, see Section 3:

Table 2 Lining pads of front disc brakes of cars in Federal Republic of Germany

Order num. q	Increasing sequence of t_q (km · 10^3)	Num. of broken parts $n_f(t_q)$	Num. of good parts $n_s(t_q)$	Middle order num. $j(t_q)$	Median order $r(t_q) = H_q$ (%)
1	8.8	2	5	2.10	1.74
2	10.3	4	5	6.53	6.02
3	10.7	3		9.85	9.24
4	11.8	1		10.96	10.31
5	12.9	2	2	13.23	12.50
6	13.4	2		15.50	14.70
7	14.4	4	1	20.09	19.14
8	15.4	4	1	24.75	23.65
9	15.6	2		27.08	25.90
10	16.4	5		32.91	31.54
11	17.7	2		35.24	33.79
12	19.3	2	2	37.65	36.13
13	21.1	6	1	45.03	43.25
14	21.6	2		47.48	45.63
15	22.4	1	1	48.74	46.85
16	23.9	1	4	50.12	48.18
17	25.3	1		51.50	49.52
18	27.7	1		52.88	50.85
19	29.1	1	1	54.30	52.23
20	29.9	5		61.40	59.09
21	30.4	2	2	64.44	62.03
22	32.1	2		67.49	64.98
23	38.4	2	2	70.81	68.19
24	39.7	3		75.78	73.00
25	40.2	3		80.76	77.82
26	40.6	3		85.74	82.63
27	41.8	2		89.06	85.84
28	45.7	2		92.38	89.05
29	50.0	3	1	98.19	94.67
30	55.7	1	1	101.09	97.48

Source: Czech Society for Quality

$$t_{11} = 0.0000, \quad t_{12} = 4.5352, \quad ..., \quad t_{1,10} = 31.1475,$$
$$t_{21} = 33.7460, \quad t_{22} = 36.5807, \quad ..., \quad t_{2,10} = 63.3745,$$
$$t_{31} = 66.3567, \quad t_{32} = 69.7614, \quad ..., \quad t_{3,10} = 113.3966.$$

For example, $\boldsymbol{t}_{32}$ expresses the length of the polygonal trail measured from the initial point 1 = (8.8, 1.74) to 22 = (32.1, 64.98). In the first group there are, in accord with previous notations, see Section

2, $n(1) = 10$ points, in the second group there are also $n(2)=10$ points, and the same $n(3)=10$ holds for the number of points in the third group. It holds that $\sum_{l=1}^{k+1} n(l) = \sum_{l=1}^{3} n(l) = 10 + 10 + 10 = 30 = n$ (the total amount of observed points).

For t we set the following knots (main and complementary, see Section 1): $T_0 = 0$, further $P = \min\{2.7460, 3.3567\} = 2.7460$, according to (3.1), hence $p = \lfloor P \rfloor = 2$. Therefore, the additional knots are

$$T_1 = \lfloor t_{1,10} \rfloor + 2 = 33,$$

$$T_2 = \lfloor t_{2,10} \rfloor + 2 = 65,$$

$$T_3 = \lfloor t_{3,10} \rfloor + 2 = 115.$$

It holds that $T_0 < T_1 < T_2 < T_3$ and $T_{l-1} \le t_{l1}$, for $l = 1, 2, 3 = k + 1$.

We choose the transformation of $t \in \langle T_0, T_{k+1} \rangle = \langle T_0, T_3 \rangle = \langle 0, 115 \rangle$ onto the interval $\langle KT_0, KT_{k+1} \rangle = \langle T'_0, T'_3 \rangle = \langle K \cdot 0, K \cdot 115 \rangle$ for the factor $K = \frac{1}{T_{k+1} - T_0} = \frac{1}{T_3 - T_0} = \frac{1}{115}$, thus onto the interval $\langle 0, 1 \rangle$. The new knots with respect to the new variable t' will then be $T'_0 = 0$, $T'_1 = 0.29$, $T'_2 = 0.57$, $T'_3 = 1$.

The program TRIO is constructed in such a way that it solves the given regression problem for a chosen $Q \in \{1, 2, 3\}$. Thus, for example, $Q = 2$ it presents the following output for the equations of the regression curve

$$x_1 = G'_1(t') = \begin{cases} 9.4844 + 21.5927t' + 30.3873(t')^2 & \text{for } 0 \le t' < 0.29, \\ 8.4484 + 28.8135t' + 17.8056(t')^2 & \text{for } 0.29 \le t' < 0.57, \\ 7.7046 + 31.4453t' + 15.4774(t')^2 & \text{for } 0.57 \le t' \le 1, \end{cases} \tag{5.1}$$

$$x_2 = G'_2(t') = \begin{cases} 1.4805 + 112.7863t' - 11.9353(t')^2 & \text{for } 0 \le t' < 0.29, \\ 0.5542 + 119.2418t' - 23.1135(t')^2 & \text{for } 0.29 \le t' < 0.57, \\ 6.5237 + 98.1193t' - 4.4982(t')^2 & \text{for } 0.57 \le t' \le 1. \end{cases} \tag{5.2}$$

It remains to determine an approximate value of the characteristic lifetime with the help of the obtained equations (5.1), (5.2). According to (5.2), there is $G'_2(0.592) = 63.0339$, $G'_2(0.595) = 63.3122$, what in turn means that $x_2 = 63.2121(\%)$ lies between these two values. In the interval (0.592, 0.595) we will search for the solution of the equation

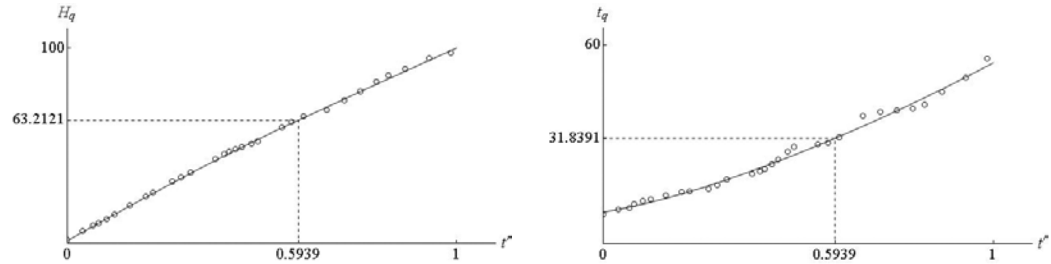
$$63.2121 = 6.5237 + 98.1193t' - 4.4982(t')^2,$$

i.e., after the rearrangement, of the quadratic equation

$$4.4982(t')^2 - 98.1193t' + 56.6884 = 0.$$

The desired solution, gained e.g. by the Bairstow iteration method, with a precision of four decimals is $t' = 0.5939$, after the substitution of which into (5.1), we get that $x_1 = 31.8391$ (km · 1 000). Therefore, the desired characteristic lifetime is $T \doteq 30\,000$ km . Figure 2 depicts the obtained solution.

Figure 2 B-spline approximation of H_q, t_q and solution for specific life expectancy



Source: Own computation

Example 5.2. The following values, in CZK · 10 000, have been obtained from the Czech Statistical Office's table Expenditures of households by level of net money income per person (ZUR0050UU) between 2006 and 2014:

Table 3 Expenditures of households by level of net money income per person 2006–2014

Year		Gross money expenditure x_1	Net money expenditure x_2	Consumption expenditure x_3
real	fictitious			
2006	0	11.5839	10.2462	9.4711
2007	1	12.9480	11.5200	10.1399
2008	2	13.3191	11.8367	10.9177
2009	3	13.5882	12.3118	11.2723
2010	4	13.6671	12.3176	11.3464
2011	5	14.3507	13.1116	11.8728
2012	6	14.1125	12.8124	11.8150
2013	7	14.3533	13.0129	12.1921
2014	8	14.7604	13.1873	12.2578

Source: Czech Statistical Office

Hypothetically, let us assume that the data for the year 2009 are missing in Table 3. For further inquiries, however, at least the probable values of x_1, x_2, x_3 are needed for that year. We try to obtain them by regression.

In $\mathbb{R}^3$ we shall, therefore, consider 8 points divided for example into 4 groups (hence $k = 3$) by 2 points:

$x_j^{(11)} = (11.5839, 10.2462, 9.4711)$, $x_j^{(12)} = (12.9480, 11.5200, 10.1399)$,

$x_j^{(21)} = (13.3191, 11.8367, 10.9177)$, $x_j^{(22)} = (13.6671, 12.3176, 11.3464)$,

$x_j^{(31)} = (14.3507, 13.1116, 11.8728)$, $x_j^{(32)} = (14.1125, 12.8124, 11.8150)$,

$x_j^{(41)} = (14.3533, 13.0129, 12.1921)$, $x_j^{(42)} = (14.7604, 13.1873, 12.2578)$, and assign to them (increasing) values of the parametric variable t:

$t_{11} = 0, \quad t_{12} = 1, \quad n(1) = 2,$

$t_{21} = 2, \quad t_{22} = 4, \quad n(2) = 2,$

$t_{31} = 5, \quad t_{32} = 6, \quad n(3) = 2,$

$t_{41} = 7, \quad t_{42} = 8, \quad n(4) = 2$ (see Table 3).

This is a case with 3 main knots, e.g. $T_1 = 2$, $T_2 = 5$, $T_3 = 7$, together with the complementary knots $T_0 = 0$, $T_4 = 8$.

For $Q = 3 = Q_3$ (cubic regression), the program TRIO provides the equations of the resulting regression curve, also the coefficients of determination $I_{x_1}^2 = 0.9859$, $I_{x_2}^2 = 0.9829$, $I_{x_3}^2 = 0.9894$, according to which 98.59% of the observed values x_1, 98.29% of the observed values x_2, and 98.94% of the observed values x_3 can be explained by this regression model. The program tells us also that the matrix $\boldsymbol{M}$ of the system of equations (2.4) does not have a dominant main diagonal.

The table of the coefficients of determination is the following:

Table 4 Coefficients of determination for linear, quadratic, and cubic regression

Regression	$I_{x_1}^2$	$I_{x_2}^2$	$I_{x_3}^2$
linear	0.9613	0.9605	0.9930
quadratic	0.9816	0.9746	0.9883
cubic	0.9859	0.9829	0.9894

Source: Own construction

It can be seen from Table 4 that the coefficients $I_{x_1}^2$, $I_{x_2}^2$ are maximal for $Q = 3 = Q_3$, while $I_{x_3}^2$ is the highest for $Q = 1 = Q_1$. Having this in mind, one can consider a kind of "optimal" regression curve for the given problem with respect to the coefficients of determination, the construction of which we in turn describe.

Generally, we shall deal with a given problem in $\mathbb{R}^m$, $m > 1$, with observed values x_j, ($j = 1, \dots, m$) by gradual application of segmented regression for $Q = 1 = Q_1$, $Q = 2 = Q_2$, and $Q = 3 = Q_3$. For a fixed $j \in \{1, \dots, m\}$, let the coefficients of determination $I_{x_j}^2$ attain their highest value for Q_r, $r \in \{1, 2, 3\}$ (the program TRIO chooses r as the lowest possible). If in (2.7) substitute x_j, for that fixed j, with the equation obtained by the particular method Q_r, in the end (for $j = 1, \dots, m$) we can comprehend (2.7) as the parametric expression of the "optimal" regression curve with respect to the coefficients of determination.

In our case, the equation of the "optimal" regression curve is

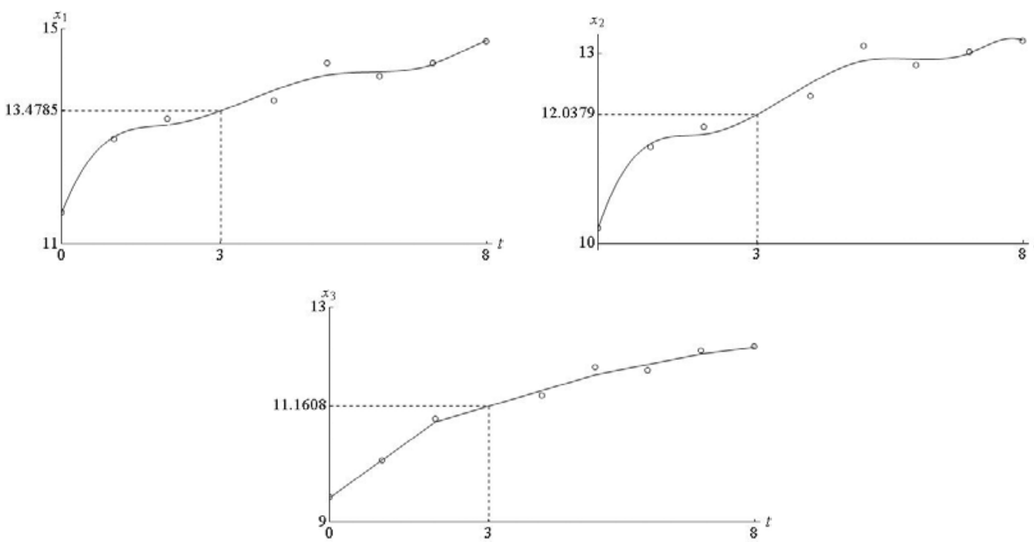
$$x_1 = \begin{cases} 11.5773 + 2.5421t - 1.3763t^2 + 0.2576t^3 & \text{for } 0 \le t < 2, \\ 13.9321 - 0.9900t + 0.3897t^2 - 0.0367t^3 & \text{for } 2 \le t < 5, \\ 0.9432 + 6.8034t - 1.1689t^2 + 0.0672t^3 & \text{for } 5 \le t < 7, \\ 72.9874 - 24.0727t + 3.2420t^2 - 0.1428t^3 & \text{for } 7 \le t \le 8, \end{cases}$$

$$x_2 = \begin{cases} 10.2390 + 2.4986t - 1.4416t^2 + 0.2819t^3 & \text{for } 0 \le t < 2, \\ 12.9217 - 1.5255t + 0.5705t^2 - 0.0534t^3 & \text{for } 2 \le t < 5, \\ -4.7048 + 9.0504t - 1.5447t^2 + 0.0876t^3 & \text{for } 5 \le t < 7, \\ 165.5307 - 63.9077t + 8.8779t^2 - 0.4087t^3 & \text{for } 7 \le t \le 8, \end{cases}$$

$$x_3 = \begin{cases} 9.4588 + 0.7058t & \text{for } 0 \le t < 2, \\ 10.2893 + 0.2905t & \text{for } 2 \le t < 5, \\ 10.7687 + 0.1947t & \text{for } 5 \le t < 7, \\ 11.2457 + 0.1265t & \text{for } 7 \le t \le 8. \end{cases}$$

For $t = 3$ we get the point (13.4785, 12.0379, 11.1608) on the optimal regression curve from these equations that we can use to substitute the hypothetically missing point in Table 3 for the year 2009, see Figure 3 also. It can be seen that this point obtained through regression lies "nearby" the actual point given in Table 3 for the year 2009.

Figure 3 Optimal regression for household expenditures (cubic for x_1 and x_2, linear for x_3)



Source: Own computation

CONCLUSION

Segmented linear, quadratic, cubic regression can be built also on cut-off splines, see (Meloun, Militký, 1994). We prefer B-splines $B_{Q,r}$, as the matrix of the system of normal equations is three-diagonal (for $Q = 1$), five-diagonal (for $Q = 2$), and seven-diagonal (for $Q = 3$), that is, its structure is much simpler than in the case of cut-off polynomials; such systems can then be solved by fast recursive methods, see (Makarov, Chlobystov, 1983). For the solution of particular exercises (see e.g. Examples 5.1 and 5.2) the computer program TRIO plays an irreplaceable role that handles every procedure leading to the final result, that is, to the equations of the regression curves.

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