Modeling of Currency Covolatilities

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Abstract

The paper deals with dynamic modeling of currency portfolios. In contrast to univariate models of exchange rates and their returns one applies multivariate time series models of the type GARCH that are capable of capturing not only conditional heteroscedasticities (i.e. volatilities) but also conditional correlations for common movements of exchange rates (so called covolatilities). One makes use of recursive estimation algorithms suggested by authors for such models which enable to control, evaluate and manage currency investment portfolios in real time. The main task of the paper is to assess whether the recursive estimation procedures suggested by the authors are applicable for real currency portfolios. It is realized by performing an extensive numerical study for bivariate portfolios of the EU currencies and US dollar concentrating on the role of the Czech crown.

Keywords	***JEL code***
Currency covolatilities, investment index, multivariate GARCH models, pay off ratio, recursive estimation	*C51, C58, F31*

INTRODUCTION

Volatility modeling plays the key role for analysis of univariate financial time series. On the other hand, understanding of comovements of more financial time series (e.g. various financial returns) is also of great practical importance since financial volatilities can move together over time across assets and markets. The models of such covolatilities are important tools for better decision-making e.g. in portfolio selection, asset pricing, hedging and risk management (see e.g. Aielli, 2013; Clements et al., 2009; Tse and Tsui, 2002).

In practice, the covolatilities are typical for currency portfolios, and one of various alternatives of their modeling consists in the application of multivariate GARCH models. In such a case one should dispose of numerically efficient estimation procedures for these models that usually contain a higher number of parameters. The aim of the paper is to assess whether the recursive estimation procedures suggested by the authors (see Section 2 for the proposed estimation methodology) are applicable just for currency portfolios. The data used in the corresponding numerical study are relatively long time series of daily exchange rates of the selected EU currencies and US dollar over eighteen years 2001–2018. In addition, special instruments assist in evaluating the management procedures based on portfolio optimization, namely the investments indices and pay off ratios (see Section 3.4). These instruments allow to draw some interesting conclusions of the case study that can have significant practical impacts.

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The paper is organized as follows. Section 1 reviews MGARCH modeling, in particular the definitions of multivariate processes MEWMA and diagonal BEKK. Section 2 presents the main principles of suggested recursive estimation of these models. Section 3 is the key one. It contains an extensive numerical study for exchange rates of EU currencies and US dollar with the objective of currency portfolio management. The last section summarizes the main conclusions of this paper.

1 MODELING OF COVOLATILITIES

The successful concept for modeling of univariate volatility consists in conditional heteroscedasticity approach realized by GARCH models. In particular, the univariate GARCH(1,1) processes $\{r_t\}$ have the form

$$r_t = \sigma_t \varepsilon_t = \sqrt{h_t}\varepsilon_t, \ \sigma_t^2 = h_t = \omega + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 , \tag{1}$$

where ε_t are *iid* random variables with zero mean and unit variance (strict white noise), σ_t^2 is the conditional variance of r_t representing the volatility at time t given information observed till time $t - 1$, and finally $\omega > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ are parameters of the process fulfilling $\alpha_1 + \beta_1 < 1$ (the sufficient conditions of positivity and stationarity of the process). One can consider processes of higher orders GARCH(r, s) or include a nonzero conditional mean but GARCH(1,1) model according to (1) is usually sufficient in routine financial applications (mainly for returns of various financial assets).

The multivariate generalization of (1) aims to model conditional covolatilities (i.e. conditional covariance matrices) in m dimensional processes $\{r_t\}$. One can apply a model scheme analogous to (1) (ignoring a potential nonzero conditional mean vector) in the form

$$\boldsymbol{r}_t = \mathbf{H}_t^{1/2} \cdot \boldsymbol{\varepsilon}_t , \tag{2}$$

where $\boldsymbol{\varepsilon}_t$ are *iid* random vectors with zero mean and identity covariance matrix and $\mathbf{H}_t$ is an $m \times m$ positive definite Ω_{t-1}-measurable matrix with square root matrix denoted as $\mathbf{H}_t^{1/2}$ (i.e. $\mathbf{H}_t = \mathbf{H}_t^{1/2}(\mathbf{H}_t^{1/2})^{\mathsf{T}}$). It represents the covariance matrix conditioned by the information observed till time $t - 1$ (Ω_t is the smallest σ-algebra such that $\{r_s\}$ is measurable for all $s \leq t$). The additional part of multivariate GARCH model is the so-called covolatility equation for the matrices $\mathbf{H}_t$ that determines the type of the corresponding model (see Sections 1.1–1.3) and contains unknown parameters ordered in a column vector $\boldsymbol{\theta}$ (therefore, one should write $\mathbf{H}_t(\boldsymbol{\theta})$ or even $\mathbf{H}_{t|t-1}(\boldsymbol{\theta})$ correctly). All vectors without transposition signs in this text are column wise.

1.1 MEWMA Models

Multivariate exponentially weighted moving average model (MEWMA or multivariate exponentially weighted moving average) is a modeling scheme which is in the univariate case supported by the commercial risk controlling software denoted as RiskMetrics (1996). In this case the covolatility equation for the conditional covariance matrix $\mathbf{H}_t$ has the form

$$\mathbf{H}_t = (1 - \lambda)\boldsymbol{r}_t \boldsymbol{r}_t^{\mathsf{T}} + \lambda \mathbf{H}_{t-1}, \ \lambda \in (0, 1) , \tag{3}$$

where λ is the only parameter to be estimated (see e.g. Hendrych and Cipra, 2019) for the univariate case. It means that the method is very parsimonious in parameters (and also constraints on λ are very simple).

1.2 Diagonal BEKK Models

Diagonal BEKK(1,1) model denoted also as dBEKK(1,1) (Baba, Engle, Kraft, Kroner, see e.g. Bauwens et al., 2006) has the covolatility equation of the form

$$\mathbf{H}_t = \mathbf{C}^{\mathrm{T}}\,\mathbf{C}^{\mathrm{T}} + \mathbf{A}\boldsymbol{r}_{t-1}\,\boldsymbol{r}_{t-1}^{\mathrm{T}}\,\mathbf{A} + \mathbf{B}\mathbf{H}_{t-1}\mathbf{B}\,, \tag{4}$$

where **C** is an $m{\times}m$ upper diagonal matrix with positive diagonal elements and **A** and **B** are $m{\times}m$ diagonal matrices with positive a_{11} and b_{11} (**A**, **B** and **C** are matrices of unknown parameters restricted to uniquely identify the given model, and the positive definiteness of $\mathbf{H}_t$ is guaranteed automatically in this model by its construction, see Bauwens et al., 2006). The model (4) is also parsimonious in parameters. Further reduction of parameters can be achieved in the scalar BEKK(1,1) model (sBEKK), where **A** and **B** are a scalar times unit matrix so that MEWMA can be obtained as a special case. Particularly, in the bivariate case one has

$$\mathbf{H}_t = \begin{pmatrix} c_{11} & 0 \\ c_{12} & c_{22} \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} r_{1,t-1}^2 & r_{1,t-1}r_{2,t-1} \\ r_{2,t-1}r_{1,t-1} & r_{2,t-1}^2 \end{pmatrix}\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} + \\ + \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}\begin{pmatrix} \sigma_{11,t-1} & \sigma_{12,t-1} \\ \sigma_{12,t-1} & \sigma_{22,t-1} \end{pmatrix}\begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \quad \text{with} \quad \begin{matrix} a_{11} > 0, & b_{11} > 0, \\ c_{11} > 0, & c_{22} > 0. \end{matrix} \tag{5}$$

1.3 Other Multivariate GARCH Models

There is a broad offer of other covolatilities models of the type MGARCH (see Bauwens et al., 2006; Silvennoinen and Teräsvirta, 2009). For instance, the constant conditional correlation GARCH(1,1) model denoted as CCC-GARCH(1,1) (see e.g. Bauwens et al., 2006) has the covolatility equation of the form

$$\mathbf{H}_t = \mathbf{D}_t\mathbf{R}\mathbf{D}_t\,, \tag{6}$$

where $\mathbf{D}_t = \mathrm{diag}\{\sqrt{h_{11,t}}, \ldots, \sqrt{h_{mm,t}}\}$ is an $m{\times}m$ diagonal matrix with diagonal elements equal to square roots of univariate volatilities $h_{ii,t}$ fulfilling univariate volatility equations:

$$h_{ii,t} = \omega_{ii} + \alpha_{ii}r_{i,t-1}^2 + \beta_{ii}h_{ii,t-1}, \quad i = 1, \ldots, m\,, \tag{7}$$

(ω_{ii}, α_{ii} and β_{ii} are parameters of the process fulfilling constraints $\omega_{ii} > 0$, $\alpha_{ii} \geq 0$, $\beta_{ii} \geq 0$, $\alpha_{ii} + \beta_{ii} < 1$) and $\mathbf{R} = (\rho_{ij})$ is an $m{\times}m$ (constant) correlation matrix (see Bauwens et al., 2006). The particular covolatilities in the matrix $\mathbf{H}_t$ can be obviously written as

$$h_{ij,t} = \rho_{ij}\sqrt{h_{ii,t}}\sqrt{h_{jj,t}}, \quad i, j = 1, \ldots, m\,, \tag{8}$$

($\rho_{ii} = 1$). A more general version of this model is the dynamic conditional correlation GARCH model (DCC-GARCH) with dynamic matrix $\mathbf{R}_t$ in (6) (see Caporin and McAleer, 2013; Engle, 2002). Another approach makes use of factor models, where the uncorrelatedness of factors can be achieved by means of various orthogonal transformations in the so-called O-GARCH (orthogonal GARCH) models.

2 RECURSIVE ESTIMATION OF MULTIVARIATE GARCH MODELS

Cipra and Hendrych (see Cipra and Hendrych, 2018; Hendrych and Cipra, 2016, 2018, 2019) developed recursive estimation for univariate GARCH models that is effective in terms of memory storage and computational complexity and can be employed e.g. in the framework of high-frequency financial time series data. The method consists in a generalization of recursive prediction error method and can be extended to the multivariate case (see Cipra, 2018) in order to construct an estimator of unknown parameters ordered in the column vector $\boldsymbol{\theta}$ by minimizing the loss function (based on the negative conditional log-likelihood criterion):

$$\min_{\theta}\sum_{t=1}^{T}[\ln|\mathbf{H}_t(\boldsymbol{\theta})|+\boldsymbol{r}_t^{\top}\mathbf{H}_t(\boldsymbol{\theta})^{-1}\boldsymbol{r}_t]. \tag{9}$$

The corresponding estimation procedure is described algorithmically by the system of the following recursive formulas:

$$\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t-1} - \eta_t\mathbf{R}_t^{-1}\mathbf{F}_t'(\hat{\boldsymbol{\theta}}_{t-1})^{\top}, \tag{10}$$

$$\mathbf{R}_t = \mathbf{R}_{t-1} + \eta_t[\tilde{\mathbf{F}}_t''(\hat{\boldsymbol{\theta}}_{t-1}) - \mathbf{R}_{t-1}], \tag{11}$$

$$\eta_t = \frac{1}{1+\xi_t/\eta_{t-1}} \text{ for } \textit{forgetting factor } \xi_t = \tilde{\xi}_t \cdot \xi_{t-1} + (1 - \tilde{\xi}),\ \xi_0 = 1,\ \tilde{\xi} \in (0, 1), \tag{12}$$

where:

$$F_t(\boldsymbol{\theta}) = \ln|\mathbf{H}_t(\boldsymbol{\theta})|+\boldsymbol{r}_t^{\top}\mathbf{H}_t(\boldsymbol{\theta})^{-1}\boldsymbol{r}_t, \tag{13}$$

$\mathbf{F}_t'(\boldsymbol{\theta})$ is the gradient of $F_t'(\boldsymbol{\theta})$, $\mathbf{F}_t''(\boldsymbol{\theta})$ is the Hessian matrix of $F_t(\boldsymbol{\theta})$ and $\tilde{\mathbf{F}}_t''(\boldsymbol{\theta})$ is the approximation of the Hessian matrix $\mathbf{F}_t''(\boldsymbol{\theta})$ such that

$$\mathbf{E}\left(\mathbf{F}_t''(\boldsymbol{\theta}) - \tilde{\mathbf{F}}_t''(\boldsymbol{\theta})|\Omega_{t-1}\right) = \mathbf{0}, \tag{14}$$

(this approximation simplifies the calculation of the corresponding Hessian matrix for particular model types). The application of forgetting factor ξ_t is typical in literature on the identification of dynamic systems since it improves the convergence properties including the statistical consistency of corresponding recursive estimators of the type (10)–(11).

For instance, let θ_k be an individual component of vector $\boldsymbol{\theta}$ of unknown parameters of diagonal BEKK(1,1) (see (4)). Then the corresponding recursive algorithm (10)–(12) for the estimation of this parameter should be supplemented by the following relations:

$$\left(\mathbf{F}_t'(\hat{\boldsymbol{\theta}}_{t-1})\right)_k = \frac{\partial F_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_k} = \left[\mathrm{tr}\left(\mathbf{H}_t^{-1}(\hat{\boldsymbol{\theta}}_{t-1})\frac{\partial\mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_k}\right) - \boldsymbol{r}_t^{\top}\mathbf{H}_t^{-1}(\hat{\boldsymbol{\theta}}_{t-1})\frac{\partial\mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_k}\mathbf{H}_t^{-1}(\hat{\boldsymbol{\theta}}_{t-1})\boldsymbol{r}_t\right], \tag{15}$$

$$\left(\mathbf{F}_t''(\hat{\boldsymbol{\theta}}_{t-1})\right)_{kl} = \frac{\partial^2 F_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_k\partial\theta_l} \approx \mathrm{tr}\left(\mathbf{H}_t^{-1}(\hat{\boldsymbol{\theta}}_{t-1})\frac{\partial\mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_k}\mathbf{H}_t^{-1}(\hat{\boldsymbol{\theta}}_{t-1})\frac{\partial\mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial\theta_l}\right), \tag{16}$$

$$\mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t) = \hat{\mathbf{C}}_t\hat{\mathbf{C}}_t^{\top} + \hat{\mathbf{A}}_t\boldsymbol{r}_t\boldsymbol{r}_t^{\top}\hat{\mathbf{A}}_t + \hat{\mathbf{B}}_t\mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})\hat{\mathbf{B}}_t. \tag{17}$$

In particular, in the bivariate model (5) with $\boldsymbol{\theta} = (c_{11}, c_{12}, c_{22}, a_{11}, a_{22}, b_{11}, b_{22})^{\top}$ the following recursive calculation of matrix differentials is possible

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial c_{11}} = \begin{pmatrix} 2\hat{c}_{11t} & \hat{c}_{12t} \\ \hat{c}_{12t} & 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial c_{11}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{18}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial c_{12}} = \begin{pmatrix} 0 & \hat{c}_{11t} \\ \hat{c}_{11t} & 2\hat{c}_{22t} \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial c_{12}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{19}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial c_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 2\hat{c}_{22t} \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial c_{22}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{20}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial a_{11}} = \begin{pmatrix} 2\hat{a}_{11t} r_{1t}^2 & \hat{a}_{22t} r_{1t} r_{2t} \\ \hat{a}_{22t} r_{1t} r_{2t} & 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial a_{11}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{21}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial a_{22}} = \begin{pmatrix} 0 & \hat{a}_{11t} r_{1t} r_{2t} \\ \hat{a}_{11t} r_{1t} r_{2t} & 2\hat{a}_{22t} r_{2t}^2 \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial a_{22}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{22}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial b_{11}} = \begin{pmatrix} 2\hat{b}_{11t} h_{11t}(\hat{\boldsymbol{\theta}}_t) & \hat{b}_{22t} h_{12t}(\hat{\boldsymbol{\theta}}_t) \\ \hat{b}_{22t} h_{12t}(\hat{\boldsymbol{\theta}}_t) & 0 \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial b_{11}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}, \tag{23}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\boldsymbol{\theta}}_t)}{\partial b_{22}} = \begin{pmatrix} 0 & \hat{b}_{11t} h_{12t}(\hat{\boldsymbol{\theta}}_t) \\ \hat{b}_{11t} h_{12t}(\hat{\boldsymbol{\theta}}_t) & 2\hat{b}_{22t} h_{22t}(\hat{\boldsymbol{\theta}}_t) \end{pmatrix} + \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix} \frac{\partial \mathbf{H}_t(\hat{\boldsymbol{\theta}}_{t-1})}{\partial b_{22}} \begin{pmatrix} \hat{b}_{11t} & 0 \\ 0 & \hat{b}_{22t} \end{pmatrix}. \tag{24}$$

The MEWMA model (see (3)) is a special case and simplifies the algorithm for the recursive estimation of the only parameter λ

$$\hat{\lambda}_t = \hat{\lambda}_{t-1} - \eta_t R_t^{-1} \left[\mathrm{tr}\left(\mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \right) - \boldsymbol{r}_t^{\top} \mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \boldsymbol{r}_t \right], \tag{25}$$

$$R_t = R_{t-1} + \eta_t \left[\mathrm{tr}\left(\mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \right) \mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} - R_{t-1} \right], \tag{26}$$

$$\mathbf{H}_{t+1}(\hat{\lambda}_t) = (1 - \hat{\lambda}_t) \boldsymbol{r}_t \boldsymbol{r}_t^{\top} + \hat{\lambda}_t \mathbf{H}_t(\hat{\lambda}_{t-1}), \tag{27}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\lambda}_t)}{\partial \lambda} = -\boldsymbol{r}_t \boldsymbol{r}_t^{\top} + \mathbf{H}_t(\hat{\lambda}_{t-1}) + \hat{\lambda}_t \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \tag{28}$$

(note the recursive calculation of the matrix differentials in (18)–(24) or in (28)).

Theoretical properties of the suggested recursive estimation algorithm coincide with the conventional non-recursive case (as t goes to infinity), where the corresponding negative conditional log-likelihood criterion is minimized. Namely, convergence and asymptotic distributional properties are identical for a sufficiently large number of observations. Refer to Ljung and Söderström (1983) for the theoretical background of the prediction error method. Simulation experiments based on these recursive formulas deliver promising results not published here. In Section 3 this approach is used in a case study for real currency data. Extensive simulations performed by the authors including theoretical aspects of recursive estimation formulas suggested by the authors for MGARCH models (and used numerically in this case study for currencies) will be published later.

3 CASE STUDY

The main objective of this contribution is to verify applicability of recursively estimated multivariate GARCH models for modeling covolatilities among currencies. Such models are capable of capturing conditional correlations in the context of various multivariate data but their efficiency should be compared. Moreover, due to recursive estimation the given approach is appropriate for financial data recorded with higher frequencies (at least daily) which is just the case of currency data. On the other hand, the conclusions of the case study presented in this section may be important from the practical point of view since one discusses the dynamic interconnection of the Czech crown with other currencies. Particularly, the estimated mutual covolatilities of corresponding currencies are indicators of the mutual currency risk that plays an important role in various regulatory systems in banking and insurance and for constructing statistical prognosis in economy (see Bollerslev, 1990).

3.1 Description of Data

More explicitly, the presented case study analyzes bivariate conditional correlation (or covariance) matrices between the log-returns of daily currency rate Czech crown/EUR against other seven currencies */EUR in EU28 and also against US dollar/EUR (see Table 1). Note that the Bulgarian lev (BGN) is omitted since it is strongly fixed to euro so that the corresponding returns in this analysis would be permanently zero. The time range of the study is Jan 2001–Dec 2018 according to ECB (refer to <http://sdw.ecb.europa.eu/browse.do?node= 9691296>), i.e. one deals with eight bivariate processes $\{\boldsymbol{r}_t\}$ of length T = 4 605 observations. Each of them has the same first component, namely the log-returns of daily currency rate Czech crown/EUR (see Table 1). The log-returns are used typically when the yield of financial assets is analyzed and compared, e.g. in our case:

$$r_t^{USD/EUR} = \ln\left(\frac{(USD/EUR)_t}{(USD/EUR)_{t-1}}\right). \tag{29}$$

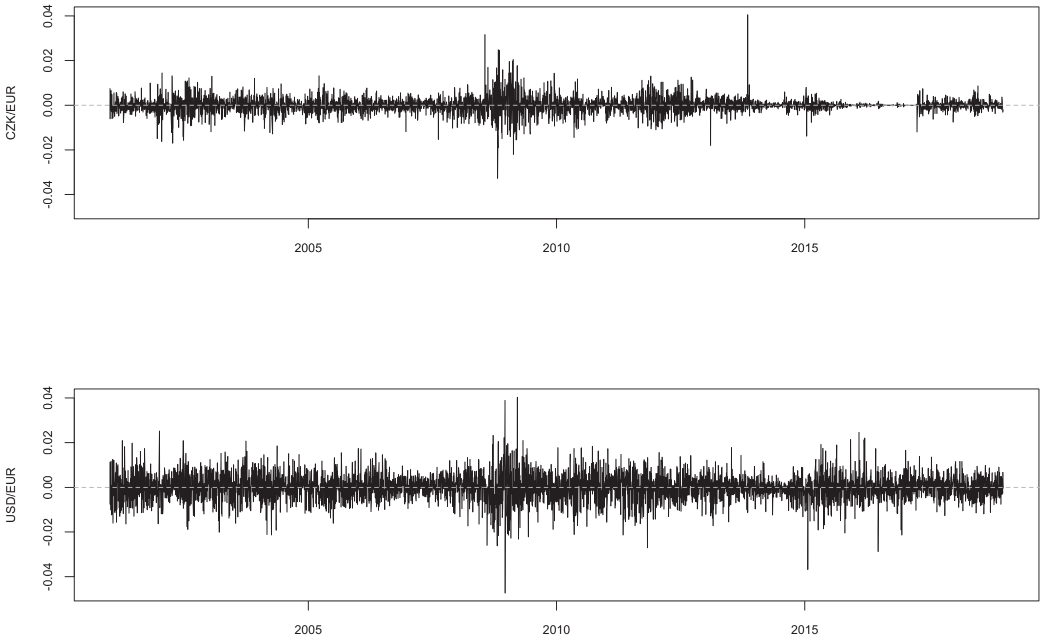
For instance, Figure 1 displays graphically log-returns of daily currency rate CZK/EUR and USD/EUR in period Jan 2001–Dec 2018.

Table 1 Currencies used in case study and components of processes $\boldsymbol{r}_t = (r_{1,t}, r_{2,t})^T$

Currency	Abbreviation	$r_{1,t}$	$r_{2,t}$
Croatian kuna	HRK	$r_t^{CZK/EUR}$	$r_t^{HRK/EUR}$
Danish krone	DKK	$r_t^{CZK/EUR}$	$r_t^{DKK/EUR}$
Hungarian forint	HUF	$r_t^{CZK/EUR}$	$r_t^{HUF/EUR}$
Polish zloty	PLN	$r_t^{CZK/EUR}$	$r_t^{PLN/EUR}$
Romanian leu	RON	$r_t^{CZK/EUR}$	$r_t^{RON/EUR}$
Swedish krona	SEK	$r_t^{CZK/EUR}$	$r_t^{SEK/EUR}$
UK pound sterling	GBP	$r_t^{CZK/EUR}$	$r_t^{GBP/EUR}$
US dollar	USD	$r_t^{CZK/EUR}$	$r_t^{USD/EUR}$

Source: Own construction

Figure 1 Log-returns of daily exchange rates CZK/EUR and USD/EUR in period JAN 2001–DEC 2018



Source: ECB <http://sdw.ecb.europa.eu/browse.do?node=9691296>

3.2 Model Estimation

For each of eight pairs of log-returns of daily currency rates $\boldsymbol{r}_t = (r_{1,t}, r_{2,t})^{\top}$ (see Table 1) one has estimated two types of the bivariate GARCH models:

- MEWMA model (see Section 1.1),
- dBEKK(1,1) model (see Section 1.2)

(such a choice is intentional since one of the objectives of our study is to compare performance of simple model schemes represented just by the model MEWMA and more complex models with larger number of parameters represented in this study by the model dBEKK(1,1)).

More precisely, we have made use of the recursive estimation algorithms of the type (15)–(24) respecting various technicalities involved when applying this procedures for real data, namely the initialization of the recursive calculations and the choice of forgetting factor in (12) (e.g. one can use a constant forgetting factor $\xi_t = 0.99$ or an increasing forgetting factor $\xi_t = 0.99\,\xi_{t-1} + 0.01$, etc., see e.g. Hendrych and Cipra, 2019). Moreover, some constraints on parameters (mainly their positivity or non-negativity) must be fulfilled to guarantee the positive definiteness of covolatility matrix $\mathbf{H}_t$.

3.3 Verification tests

The particular models are verified by means of the bivariate Ljung-Box test (see Cipra, 2013, p. 348; Ljung and Box, 1978; Lütkepohl, 2005, p. 171). The test is applied in two versions: the first version (see the test statistics Q(MEWMA) and Q(dBEKK) in Table 2) based on estimated bivariate residuals verifying the serial uncorrelatedness and the second version (see test statistics $Q2$(MEWMA) and $Q2$(dBEKK) in Table 2) based on squares of estimated bivariate residuals verifying the elimination of heteroscedasticity. The maximum delays of corresponding statistics Q is chosen as 8 ~ ln(4 605) (i.e. the natural logarithm of time series lengths as it is usually recommended).

Table 2 presents p-values for both versions of this test. In both cases, the better results are obtained for MEWMA model (the residual uncorrelatedness is rejected on the level of significance 5% only in one case for Q(MEWMA) and in no case for Q2(MEWMA)). The substantially worse results for Q(dBEKK) and Q2(dBEKK) consist mainly in the fact that estimation of models with higher number of parameters is a more complex problem in the volatile environment of currency exchange rates (it is also confirmed by other empirical results for currencies which are not reported in this paper).

Table 2 Multivariate Ljung-Box tests (columns $Q(\cdot)$ and $Q2(\cdot)$ contain p-values)

Exchange rate 1	Exchange rate 2	Q(MEWMA)	Q2(MEWMA)	Q(dBEKK)	Q2(dBEKK)
CZK/EUR	DKK/EUR	0.36637	0.21492	0.00554	0.49086
CZK/EUR	GBP/EUR	0.00825	0.57121	0.65633	0.00000
CZK/EUR	HRK/EUR	0.36117	1.00000	0.00049	0.00111
CZK/EUR	HUF/EUR	0.22317	0.95423	0.60785	0.45449
CZK/EUR	PLN/EUR	0.16215	0.08995	0.05619	0.03928
CZK/EUR	RON/EUR	0.22871	0.76751	0.00921	0.04658
CZK/EUR	SEK/EUR	0.54316	0.37302	0.00145	0.59162
CZK/EUR	USD/EUR	0.49079	0.99968	0.47606	0.55483
Number of non-rejections of the null		7	8	4	4

Source: Own construction

There are other tests that could be applied in our study. In particular, Laurent et al. (2012) addressed the issue of the selection of MGARCH models in terms of variance matrix of forecasting accuracy with a particular focus on relatively large-scale problems. They considered 10 assets from the New York Stock Exchange and compared various models based on 1-, 5- and 20-day-ahead conditional variance forecasts over a period of 10 years using the so-called model confidence sets (MCS) and the superior predictive ability (SPA) tests.

3.4 Outputs

Figure 2 presents conditional correlations among daily log returns of exchange rates CZK/EUR and */EUR for seven currencies from Table 1 recursively estimated by means of models MEWMA and dBEKK(1,1) in period Jan 2001–Dec 2018. One can see that the used model approach injects an important dynamic aspect to the analysis of mutual behavior of currencies (sometimes with very intensive covolatilities). Some exchange rates show significant correlation links to the Czech crown. In particular, one can notice the relatively high positive correlations between the Czech crown/EUR against the Hungarian forint/EUR or the Polish zloty/EUR. On the contrary, the Czech crown/EUR is nearly neutral against the Croatian kuna/EUR or the Danish krone/EUR (if we ignore the presence of noise). There is another interesting aspect, namely the correlation results obtained by means of various models can differ significantly.

For instance, the conditional correlations by the MEWMA seem to be in some periods more volatile than by the dBEKK(1,1) (refer to the Czech crown/EUR against the Hungarian forint/EUR since 2015) or can even show some trends (consult the increasing trend of Czech crown/EUR against the UK pound sterling/EUR in the period 2013–2016).

Further results in Figure 3 correspond to another objective of our study, namely to evaluate the applicability of MGARCH models in the portfolio optimization context. A similar approach to the currency portfolio investment has been suggested by Franke and Klein (1999) for optimal currency portfolio management using neural networks. One considers eight currency portfolios $\boldsymbol{r}_t = (r_{1,t}\,, r_{2,t})^{\mathsf{T}}$ from Table 1 that are optimized each trading date. More explicitly, one recalculates weights for minimum variance portfolios recursively in time applying one-step-ahead predictions of conditional covolatilities $\mathbf{H}_t = \mathbf{H}_{t|\,t-1}$ obtained by the corresponding models MGARCH recursively estimated for time t using information known till time $t-1$, see Section 2 (gains like interests and losses like transaction costs caused by managing the portfolio are neglected). In other words, one reevaluates each trading date the portfolio weights $\mathbf{w}_t = (w_t, 1 - w_t)^{\mathsf{T}}$ when minimizing the portfolio risk measured by the portfolio variance (or by its standard deviation), i.e. one solves optimization problems:

$$\min \mathbf{w}_t^{\mathsf{T}}\, \mathbf{H}_t \mathbf{w}_t \qquad \text{subject to } \mathbf{w}_t^{\mathsf{T}}\, \mathbf{1} = 1,\, \mathbf{w}_t \geq 0 \tag{30}$$

Figure 2 Conditional correlations between daily log returns of exchange rates CZK/EUR and */EUR for eight currencies from Table 1 recursively estimated by means of models MEWMA and dBEKK(1,1) in period JAN 2001–DEC 2018

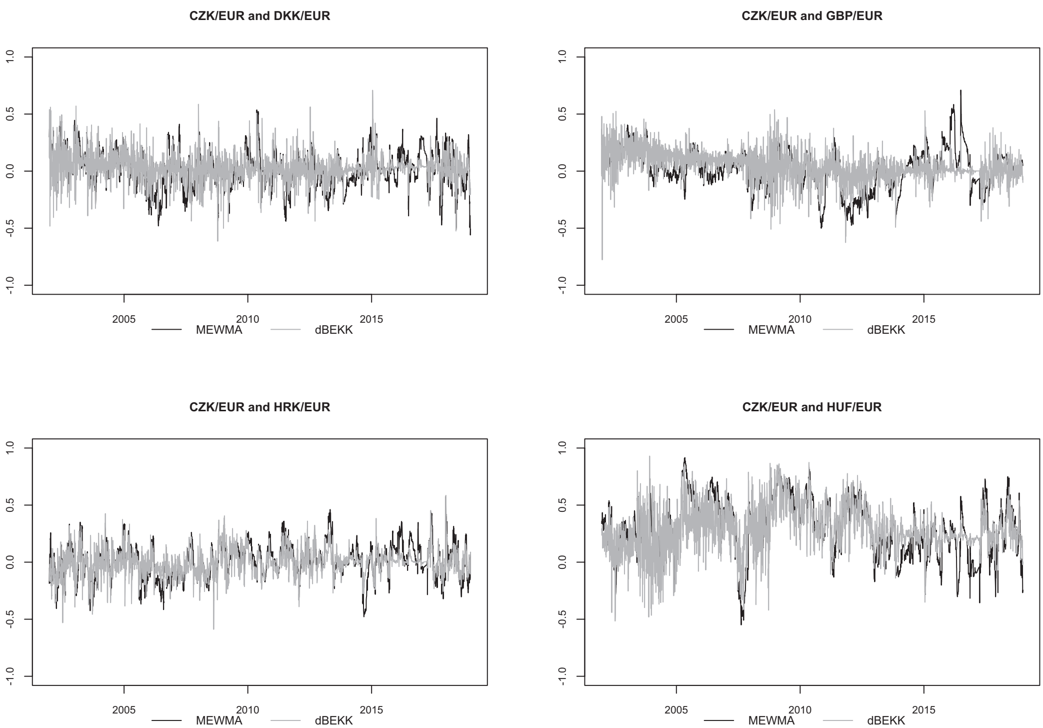
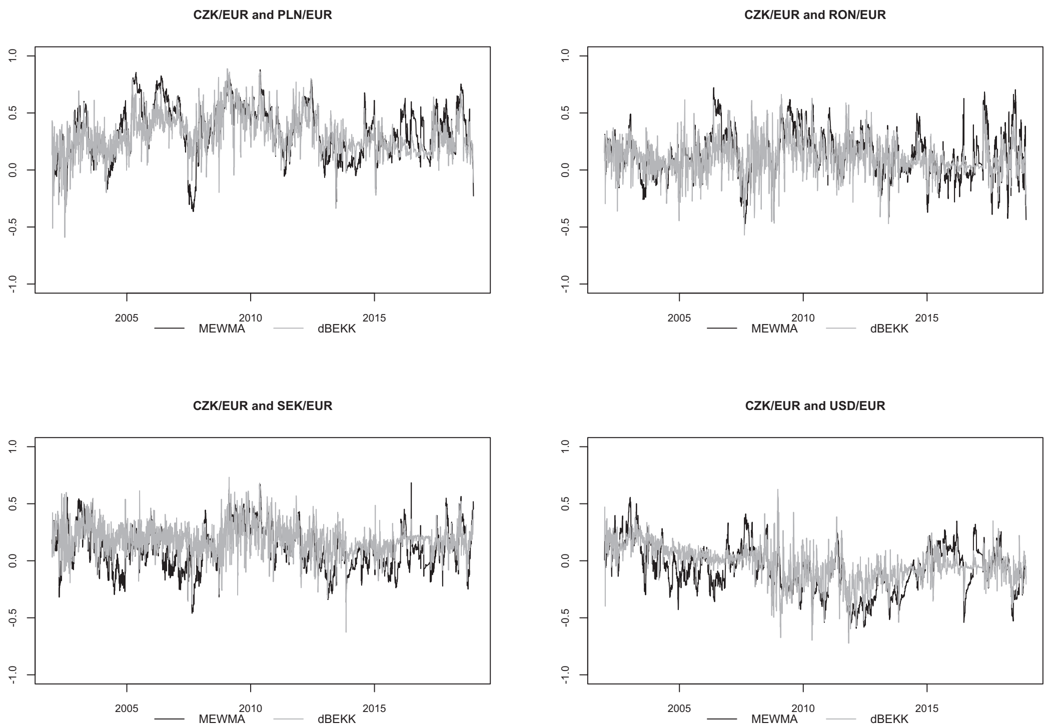


Figure 2 (continuation)



Source: Own construction

for unit vectors **1**. The minimization of portfolio risk is a reasonable criterion for daily management of currency portfolios since the average daily yield is usually near to zero. One could also permit the short sales operations, when the non negativity constraints for weights in (30) are canceled which allows an explicit solution of (30) in the form

$$\mathbf{w}_t = \frac{\mathbf{H}_t^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{H}_t^{-1}\mathbf{1}}. \tag{31}$$

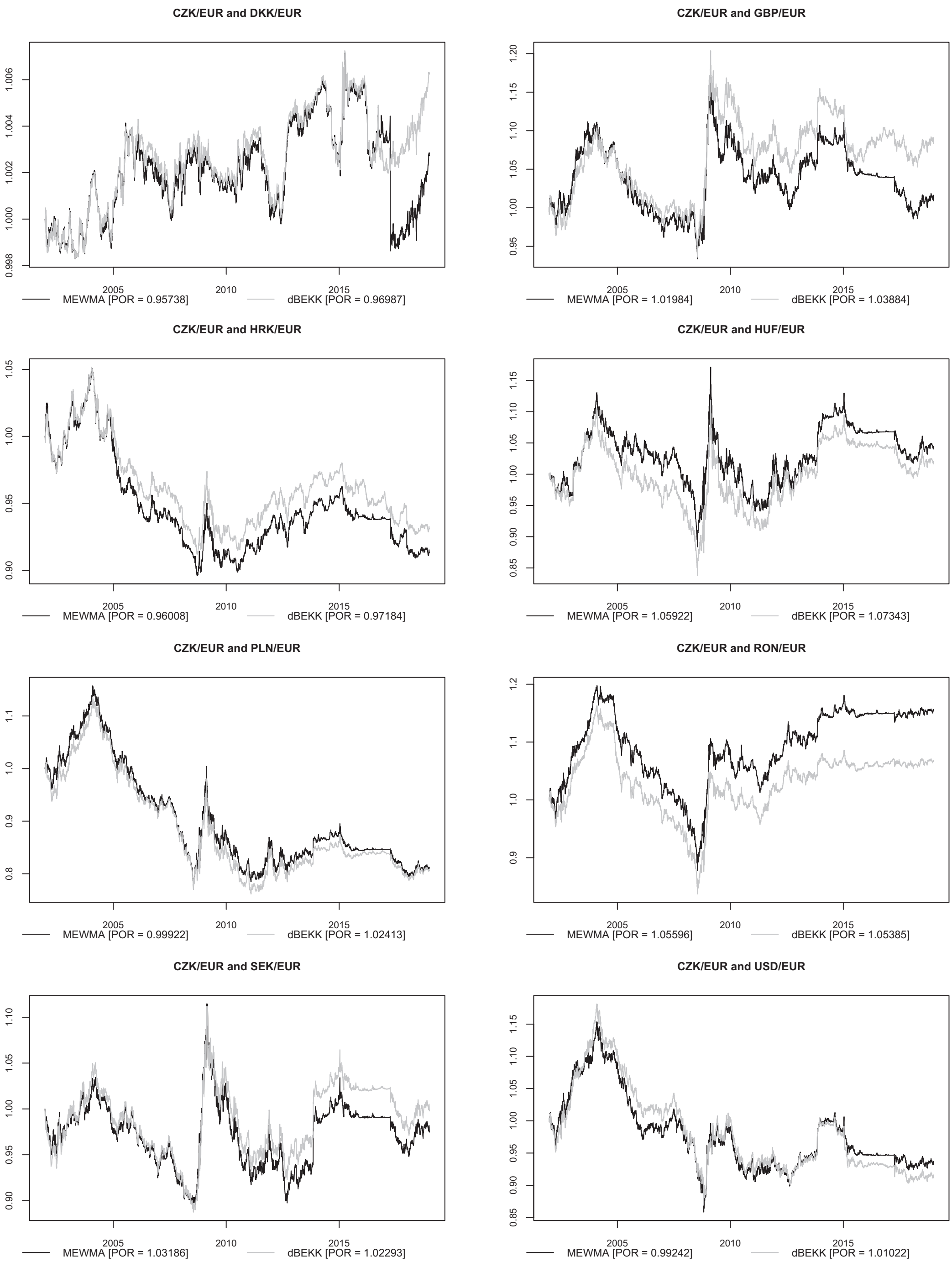
The described management procedure has two numerical outputs:

Firstly, one follows the development of an initial investment in amount of 1 EUR for each portfolio. Figure 3 shows for each pair of currencies from Table 1 the plot of investment indices $\{I_t\}$ defined in time t recursively as

$$I_t = I_{t-1}\{w_t \exp(r_{1,t}) + (1 - w_t) \exp(r_{2,t})\}, t = 2, 3, ..., T \tag{32}$$

with I_1 = 1 EUR. In other words, I_t represents the value (in EUR) at time t of particular portfolio composed of CZK and corresponding currency from Table 1 with initial investment capital

Figure 3 Investment indices for eight exchange rates from Table 1 recursively estimated by means of models MEWMA and dBEKK(1,1) in period JAN 2001–DEC 2018



Source: Own construction

of 1 EUR when the optimal weights are always constructed one-step-ahead in MEWMA or dBEKK(1,1) model by solving the optimization task (30) repeatedly.

We can see that the investment strategy based on MEWMA model guarantees better investment results than dBEKK(1,1) model (with exception of the portfolio investing to CZK and DKK and the portfolio investing to CZK and GBP where both models give nearly identical results). In particular, the best investment results in terms of the investment index were obtained by investing the initial capital to CZK and GBP followed by investing to CZK and USD.

Secondly, for each plot in Figure 3 one calculates so-called pay off ratio *POR* (see Franke and Klein, 1999):

$$POR = \frac{\frac{1}{T_+}\sum_{t=2}^{T}\{w_t[\exp(r_{1,t})-1]+(1-w_t)[\exp(r_{2,t})-1]\}^+}{\frac{1}{T_-}\sum_{t=2}^{T}\{w_t[\exp(r_{1,t})-1]+(1-w_t)[\exp(r_{2,t})-1]\}^-}, \tag{33}$$

where $a^+ = \max(a, 0)$ is the positive part of a, $a^- = \max(-a, 0)$ is the negative part of a, T_+ is the number of nonzero summands in the numerator of (33) and T_- is the number of nonzero summands in the denominator of (33). The optimal portfolio weights w_t and $1 - w_t$ are again constructed one-step-ahead using MEWMA or dBEKK(1,1) model similarly as in (32), i.e. by solving the optimization task (30) repeatedly. Obviously, if $POR > 1$ (or $POR < 1$) then the mean of positive returns in the given investment portfolio is larger (or smaller) than the mean of negative returns, respectively.

For our investment portfolios the pay off ratios are slightly above the equilibrium unit value (with exception of the portfolio investing to CZK and DKK which is anomalous also from the point of view of investment indices, see above). This result is not surprising due to positive correlations between components of particular investment portfolios (see above) so that it is difficult to enforce more significant daily diversification effects.

CONCLUSION

The realized case study confirmed that the application of multivariate GARCH models is an approach that can be useful when one models and controls on line portfolios of assets with a higher frequency of records and potential conditional correlations among components changing in time. The typical examples are currency portfolios examined in our case study. The study found various dynamical links among some currencies and outlined potential applications in portfolio management.

The study showed that simpler models with lower number of parameters should be preferred in practice. Particularly, in our case the models MEWMA seem to be much better than more sophisticated models dBEKK both in the sense of statistical tests and in the sense investment results based on the corresponding model approach.

Future research can extend the dimension of models or even include portfolios with a high number of components which is the case of stock indices.

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