Income and Inequality Measures in Households in the Czech Republic and Poland based on Zenga Distribution

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Abstract

Income is one of the appropriate indicators for the evaluating of living standards of people, so it is important to examine the distribution of income and its degree of differentiation in society. A lot of research has been directed at describing empirical distributions by the theoretical model. In 2010 Zenga proposed a new three-parameter model for economic size distribution. The aim of this paper was to apply Zenga model to income distribution in the Czech Republic and in Poland. The results of conducted approximations obtained by means of D'Addario's invariants methods confirm that the Zenga distribution is a good income distribution model which can be successfully applied to income inequality analysis and income distribution comparisons both in the Czech Republic and in Poland.

Keywords	***DOI***	***JEL code***
Household income, Zenga distribution, Gini inequality index, Zenga income inequality index	*https://doi.org/10.54694/stat.2021.12*	*C10, C13, C15*

INTRODUCTION

Since Pareto proposed his first income distribution model, many economists and mathematicians tried to describe empirical distributions by simple mathematical formulas with a small number of parameters. For a majority of countries we observe similar, characteristic shape of the income distribution. For many countries income distributions are single-modal with right-sided asymmetry. Theoretical models can be used for the approximation of empirical distributions, in order to obtain information from the data by the parameter estimates and their interpretation. After the one parameter Pareto model (Pareto, 1987), the two-parameter models such as the log-normal model (Gibrat, 1931), the gamma (Salem and Mount,

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1974), and the Weibull model (Bartels and Van Metele, 1976) were introduced. The three-parameter models appeared, such as the generalized gamma (Taille, 1981), Singh-Maddala (Singh and Maddala, 1976) and Dagum (Dagum, 1977). The four-parameter models called the generalized beta of the first and second kind (GB1 and GB2) were introduced by McDonald (1984). Descriptions of many income distributions and approximation methods are included in the papers (Kleiber and Kotz, 2003; Brzeziński, 2013). For many countries the three-parameter distributions have represented a good approximation of income distribution (Ćwiek and Ulman 2019).

In this paper a new three-parameter model for distributions by size proposed by Zenga (2010) is analyzed. Zenga distribution is a Beta mixture defined for non-negative distributions, that has positive skewness and Paretian right tail. Zenga model is particularly indicated for describing income, financial and actuarial distributions. The parameters of Zenga distribution separately control the location and inequality. Zenga distribution is obtained as a mixture of Poliscchio distributions with mixing function given by the Beta density with μ constant and $k \in (0,1)$. Zenga model has three parameters: is a scale parameter and it is equal to the expected value, and are shape parameters that inequality depends on. This has implications in parameter estimation, because the restrictions on the expected value and on inequality measure can be imposed separately (Arcagni and Porro 2013). The estimate parameters of Zenga distribution can be found, through D'Addario's invariants method or by imposing restrictions on numerical optimization. These methods have been applied by (Zenga et al., 2010b) and (Arcagni, 2011) to estimate the parameters of Zenga distribution. Due to the Paretian right-tail, Zenga distribution can fit the income distributions for large values. This model can embrace several shapes and this feature allows a good fitting also for small incomes. The model can be zero modal and unimodal although it has only two shape parameters and the expected value of Zenga model is always finite. This is an advantage, because if the expected value of fitted model is not finite, it does not have economic interpretation. Studies performed in various countries show that Zenga distribution exhibit high conformance to empirical distributions of incomes, (Zenga et al., 2010b; Arcagni and Porro, 2013).

Various theoretical distributions were used to approximate the income distributions of the Czech Republic and Polish population (Bartošová and Bína, 2009; Bílková, 2008; Duspivá and Spåcil, 2011; Malá, 2011; Malá, 2013; Matějka and Duspivová, 2013; Kordos, 1990; Brzeziński, 2013; Ostasiewicz, 2013; Salamaga, 2016; Jędrzejczak and Pekasiewicz, 2018; Ćwiek and Ulman 2019). Zenga probability distribution has not been used so far to analyze income distributions in the Czech Republic. The density function of the Zenga has a number of interesting statistical properties and thus easily adapts to various types of empirical income distributions. The aim of this paper is to apply the Zenga distribution to the analysis of income distributions of households in Czech Republic and Poland, which have been distinguished by means of the main or leading source of maintenance.

This paper is organized as follows. Zenga income distribution is presented in Section 1 providing probability density function, distribution function and some other main features. Section 2 is devoted to the description of the Lorenz curve and the Zenga curve. The application of Zenga model to empirical income distributions for the Czech Republic and Poland is shown in Section 3. In the same section, an analysis of income inequality in both countries was presented. In the last section some final remarks are discussed.

1 ZENGA INCOME DISTRIBUTION

The three-parameter income model was introduced by Zenga (Zenga, 2010a; Zenga, Pasquazzi, Polisicchio, 2010b; Zenga, Pasquazzi, Zenga, 2012). The density function $f(x:\mu;\alpha;\theta)$ in Zenga probability distribution has been obtained as a mixture of Poliscichio's (Poliscichio, 2008) following truncated Pareto density:

$$v(x:\mu;k)=\begin{cases}\frac{\sqrt{\mu}}{2}k^{0.5}(1-k)^{-1}x^{-1.5}, & \mu k\le x\le\frac{\mu}{k};\quad \mu>0,\quad 0<k<1,\\ 0, & \text{otherwise,}\end{cases} \tag{1}$$

with a fixed $\mu > 0$ and all the values of k in the interval (0; 1). The density on the parameter k is given by the beta density and has the following form:

$$g(k:\alpha;\theta)=\begin{cases}\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}, & 0<k<1;\quad \theta>0,\ \alpha>0,\\ 0, & \text{otherwise},\end{cases} \tag{2}$$

where $B(\alpha;\theta$ is the beta function.

The model is characterized by the probability density function $f(x:\mu;\alpha;\theta)$ for non-negative variables:

$$f(x:\mu;\alpha;\theta)=\int_0^1 v(x:\mu;k)\,g(k:\alpha;\theta)\,dk$$
$$=\begin{cases}\frac{1}{2\mu B(\alpha;\theta)}\left(\frac{x}{\mu}\right)^{-1.5}\int_0^{\frac{x}{\mu}}k^{\alpha-0.5}(1-k)^{\theta-2}\,dk, & \text{for}\quad 0<x<\mu,\\ \frac{1}{2\mu B(\alpha;\theta)}\left(\frac{\mu}{x}\right)^{1.5}\int_0^{\frac{\mu}{x}}k^{\alpha-0.5}(1-k)^{\theta-2}\,dk, & \text{for}\quad x>\mu.\end{cases} \tag{3}$$

The parameter α is an inverse inequality indicator and θ is direct inequality indicator. In particular the bigger value of the parameter α the less unequal the distribution and the bigger the value, the more unequal the distribution (Porro, 2015).

In the case $\theta > 0$, the distribution function $\mathrm{F}(x:\mu;\alpha;\theta)$ is described by the formula:

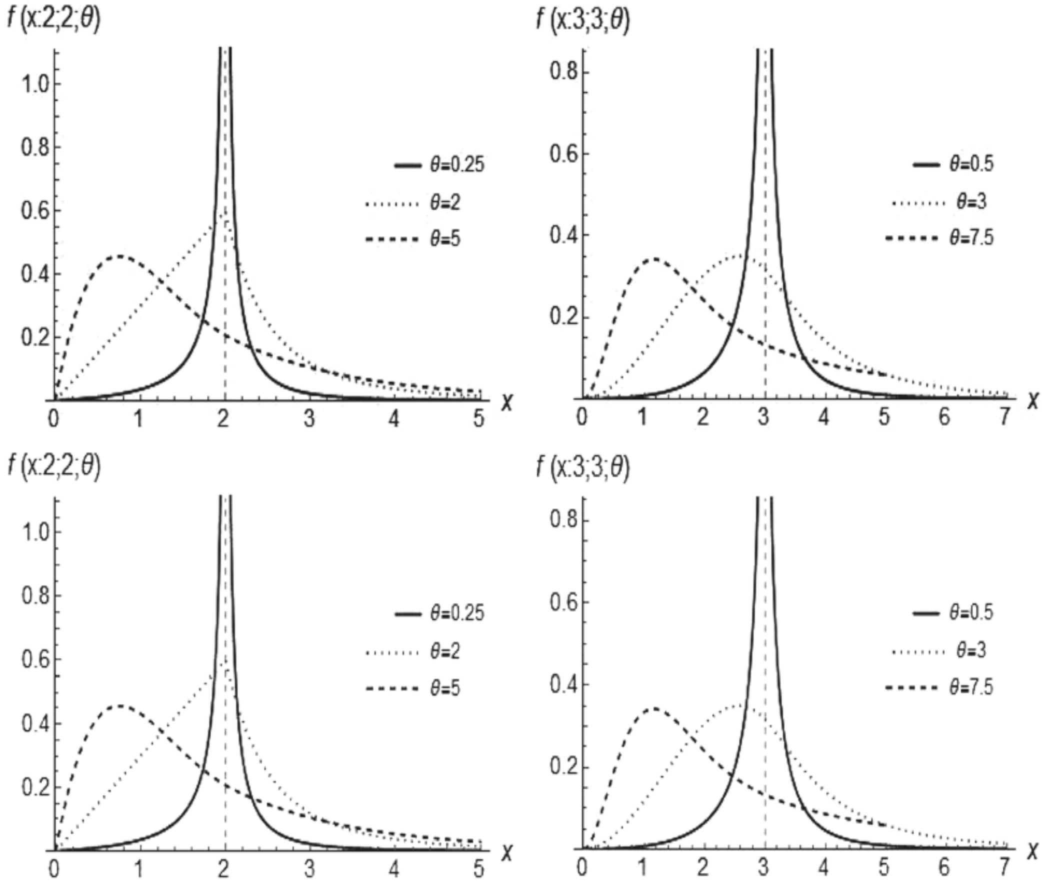
$$F(x:\mu;\alpha;\theta)=\begin{cases}\frac{1}{B(\alpha,\theta)}\sum_{i=1}^{\infty}\left\{IB\left(\frac{x}{\mu};\alpha+i-1;\theta\right)-\left(\frac{\mu}{x}\right)^{0.5}IB\left(\frac{x}{\mu};\alpha+i-0.5;\theta\right)\right\}, & \text{if}\ 0<x\le\mu\\ 1-\frac{1}{B(\alpha,\theta)}\sum_{i=1}^{\infty}\left\{\left(\frac{\mu}{x}\right)^{0.5}IB\left(\frac{\mu}{x};\alpha+i-0.5;\theta\right)-IB\left(\frac{\mu}{x};\alpha+i;\theta\right)\right\}, & \text{if}\ \mu<x.\end{cases} \tag{4}$$

where:

$$IB(x;\alpha;\theta)=\int_0^x t^{\alpha-1}(1-t)^{\theta-1}\,dt,\quad 0<x<1, \tag{5}$$

is the incomplete beta function.

Figure 1 Density plots of the Zenga income model for some parameter values



Source: Authors' calculations

It is easy to see that density functions in Figure 1 and distribution functions in Figure 2 are very flexible and its plot can have different shapes depending on the values of the parameters. The Zenga density function and distribution function allow for a wider variety of shapes than traditional three-parameter models of income distributions.

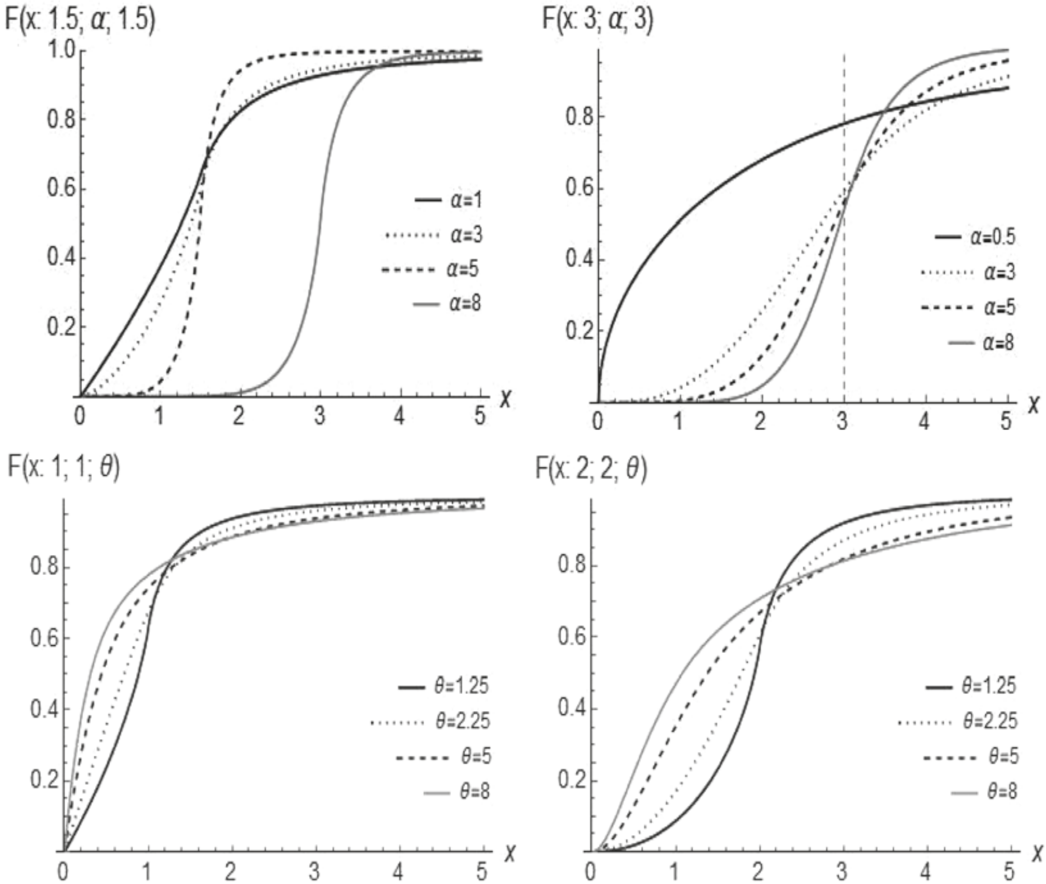
The Zenga income distribution meets the criteria set of the theoretical income distribution. The most important are the following:

- Convergence to the Pareto law for high income groups.
- Existence of only a small number of finite moments of a distributions (heavy tails).
- Goodness of fit for a whole range of a distribution.
- Simple interpretation of parameters.
- Simplicity (or small number of parameters).

2 POINT INEQUALITY CURVES

The analysis and the graphical representation of inequality play a very important role in economics. Inequality curves are graphical methods used to analyse, characterize and to generally relate to inequality indexes. The Lorenz curve (Lorenz, 1905) and the Zenga curve (Zenga, 2007a) are the most used inequality

Figure 2 Distribution function plots of the Zenga income model for some parameter values



Source: Authors' calculations

curves these days. The Zenga curve (Zenga, 2007a, 2007b) can embrace different shapes, which allow to distinguish different situations in terms of inequality. We obtain for the Zenga density function $f\left(x:\mu;\alpha;\theta\right)$ the graphs of the Lorenz curve and of Zenga's point inequality measure (Zenga, 2007a). These point measures are invariant to scale transformation, so it is enough to consider this case $f\left(x:1;\alpha;\theta\right)$. To obtain the graphs of these point measures we need to evaluate the first incomplete moment $H\,(x)$.

The first incomplete moment of the density $f\left(x:\mu;\alpha;\theta\right)$ is given by:

$$H(x\colon 1;\alpha;\theta)=\begin{cases} H_1(x\colon 1;\alpha;\theta)=\dfrac{1}{B(\alpha;\theta)}\sum\limits_{i=1}^{\infty}[x^{0.5}IB(x\colon \alpha+i-0.5;\theta)+\\ \qquad -IB(x\colon \alpha+i;\theta)],\\ \qquad \text{if } 0<x\leq 1\\ H_2(x\colon 1;\alpha;\theta)=1-\dfrac{1}{B(\alpha;\theta)}\sum\limits_{i=1}^{\infty}[IB\,(\dfrac{1}{x}\colon \alpha+i-1;\theta)+\\ \qquad +x^{0.5}IB\,(\dfrac{1}{x}\colon \alpha+i-0.5;\theta)],\\ \qquad \text{if } 1<x. \end{cases} \tag{6}$$

The graphs of the Lorenz curve can be obtained putting in abscissa $F(x)$ and in ordinate $H(x)$ for a sufficient number of values of x. Zenga's point inequality (Zenga, 2007a) has the form:

$$A(x) = \frac{F(x:1;\alpha;\theta) - H(x:1;\alpha;\theta)}{F(x:1;\alpha;\theta)\left[1 - H(x:1;\alpha;\theta)\right]}.$$

The graphs of Zenga's point inequality can be obtained putting in abscissa $\mathrm{F}(x:1;\alpha;\theta)$ and $A(x)$ on the ordinate axis for a sufficient number of values of x.

Figures 8 to 12 contain the graphs of the Lorenz curve L(p) and the Zenga curve $I(p)$-where $p = F(x)$.

3 ANALYSIS OF INCOMES

In this paper all calculations are based on the data from research European Quality of Life Surveys (EQLS), whose purpose is to measure both objective and subjective indicators of the standard of living of citizens and their households. The analyzed data of monthly household income has been recalculated into the income per member of the household expressed in Euro. The Zenga model for the Czech Republic and Poland was used for two periods: 2007 and 2016.

The results of the approximation of the empirical income distributions in the Czech Republic and Poland by means of the Zenga income model, together with the goodness of fit measures $A_{1,}$ and W_p and estimated standard errors of point estimates are presented in Table 1 and in Figures 3–6. The approximation methods for the Zenga income distribution have been analysed in paper (Zenga, 2010a; Arcagni and Porro, 2013). Table 1 contains the parameter estimates obtained by means of D'Addario's invariants methods which gave the best fitting.

To find a degree of adjustments of a theoretical distribution to the empirical one, we have calculated the goodness of fit measures: the Mortara index $A_{1,}$:

$$A_1 = \frac{1}{n}\sum_{j=1}^{s}\left|n_j - \hat{n}_j\right|, \tag{7}$$

where n_j and $\hat{n}_j$ are respectively the observed and the estimated frequencies of the j-th interval. The n observations are grouped into s = 10 intervals, where $j = 1, ..., s$. The smaller the value of , A_1 the higher the consistency of compared distributions.

The coefficient of distributions similarity W_p for the empirical data arranged into a grouped frequency distribution with s-class intervals can be calculated by the formula:

$$W_p = \sum_{i=1}^{s} min(w_i\ ; w_i^{'}), \tag{8}$$

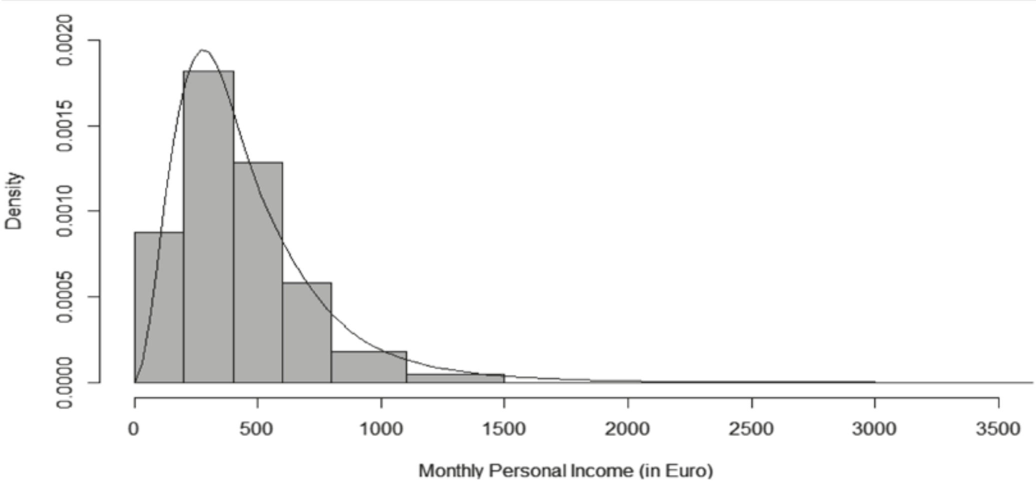
where w_i and $w_i^{'}$ represent empirical and theoretical frequencies, respectively. The bigger the value of W_p, the higher the consistency of compared distributions. The presented measures of distributions similarity has a clear interpretation. The data aggregation was necessary to calculate indexes of goodness of fit. Aggregation was applied using methods presented in the papers (Zenga, 2010a; Porro, 2015; Brzeziński, 2013). In order to obtain personal income distributions, in all our estimations we have to recalculated original data. Observations with zero incomes and the data gaps were excluded from the analysis, but this affected less than 1.5% of all observations for all of our data sets. It is worth noting here that the removed observations are few in the scale of the entire sample. Moreover, it is necessary to apply the Zenga income model. Note that the Zenga probability density function is defined for non-negative income. This approach can be useful for many reasons. Firstly, applying the theoretical income model simplifies the analysis. Knowledge of the model of income distribution, which is a simple approximation of empirical distribution and knowledge of tendency of its parameters development may be used to predict behavior of the particular variable in the following period of time. Moreover, the approximation of the empirical wage and income distributions by means of the theoretical curves can smooth the irregularities coming from the method of data collecting. Table 1 presents estimation results for income distributions based on the data coming from EQLS. The table contains the parameters estimates obtained by means of D'Addario's invariants method, as it is described in (D'Addario, 1939; Zenga, 2010). The distribution of estimators has been analyzed by parametric Bootstrap resampling. Numerical methods of optimization and Bootstrap resampling were carried out using Mathematica program.

Table 1 Estimation results for income distributions in the Czech Republic and Poland based on the data coming from EQLS

Country	Year	Estimated values of parameters and standard errors			Indexes of goodness of fit	
		$\hat{\mu}$	$\hat{\alpha}$	$\hat{\theta}$	A_1	W_p
Czech Republic	2007	472.2562 (10.7448)	2.8870 (1.1222)	4.5943 (1.9652)	0.1044	0.9478
Poland	2007	309.9459 (8.4357)	1.5561 (0.0816)	3.4597 (0.0929)	0.0661	0.9668
Czech Republic	2016	822.0742 (16.6443)	2.0000 (0.1364)	3.0000 (0.2848)	0.1397	0.9309
Poland	2016	553.4071 (22.4882)	1.5183 (0.3439)	3.5151 (1.0528)	0.0796	0.9595

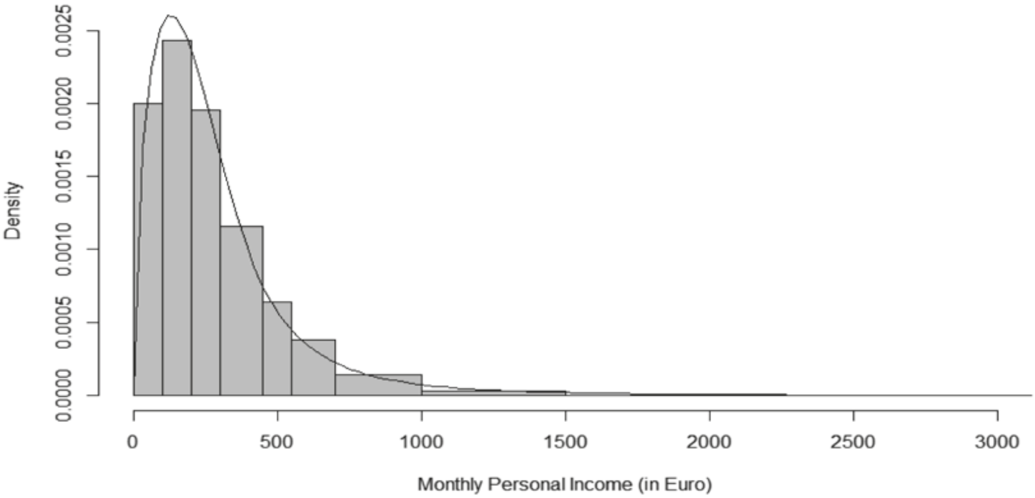
Note: Standard errors are given in parentheses.
Source: Author's calculation

Figure 3 Zenga density fitted to empirical the Czech Republic household income distribution in the year 2007 (787 observations)



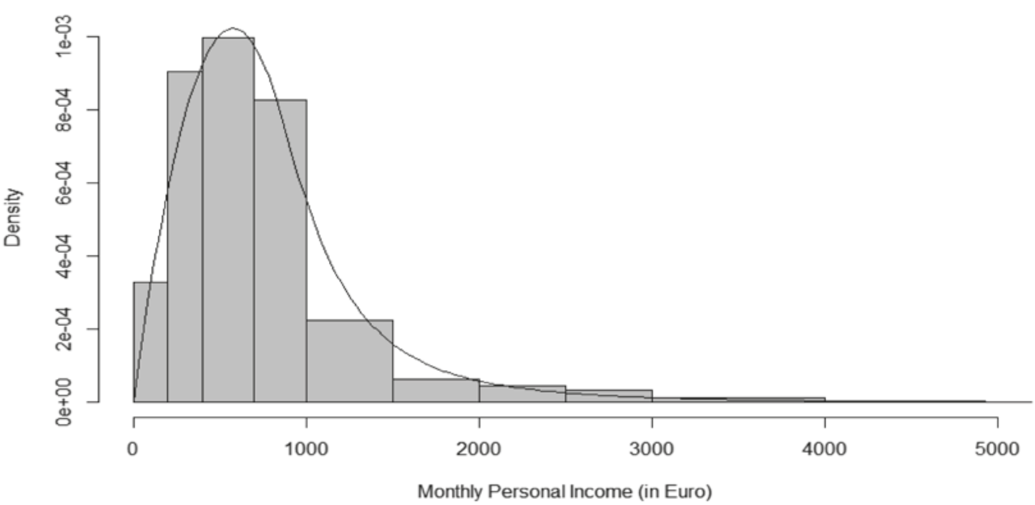
Source: Authors' calculation

Figure 4 Zenga density fitted to empirical Poland household income distribution in the year 2007 (1 065 observations)



Source: Authors' calculation

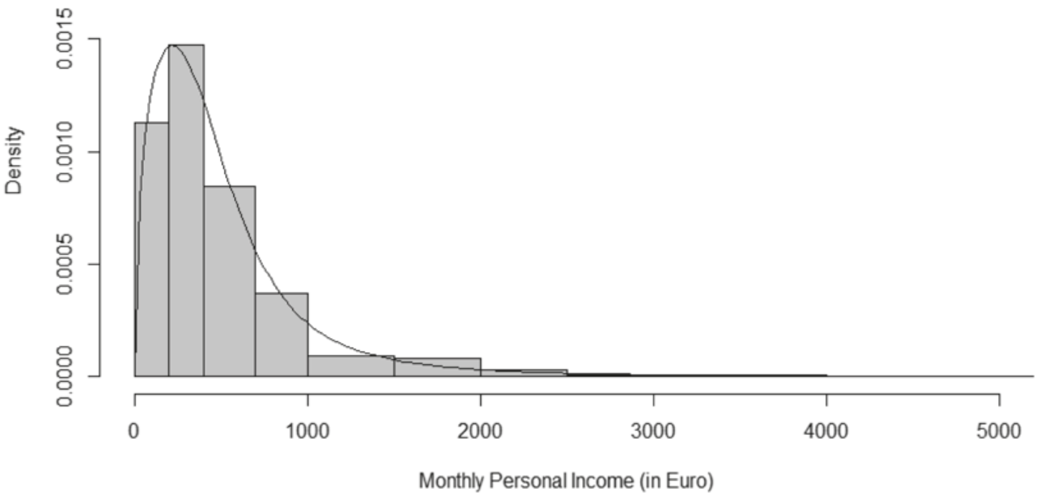
Figure 5 Zenga density fitted to empirical the Czech Republic household income distribution in the year 2016 (686 observations)



Source: Authors' calculation

The Zenga income model is characterized by the high flexibility. The results of the calculations presented in Table 1 and in Figures 3–6 confirm good consistency of the Zenga probability distribution with the empirical income distribution in the Czech Republic and Poland. Which is consistent with our previous work (Jędrzejczak and Trzcińska, 2018; Trzcińska, 2020). The consistency is slightly worse for the Czech Republic and Poland in the year 2016. Note that in 2016 the distributions for both countries are more asymmetrical (Figures 5 and 6). The precision of the estimation is presented in Table 2.

Figure 6 Zenga density fitted to empirical Poland 2016 household income distribution in the year 2016 (733 observations)



Source: Authors' calculation

Table 2 Empirical and fitted characteristics for the Czech Republic and Poland

Country	Year	Mean	Median	Cumulative value for the mean	Variance	Quantile 0.1	Quantile 0.25	Quantile 0.75	Quantile 0.95
Empirical values									
Czech Republic	2007	472.2562	376.3158	0.6404	143 554.2	153.5088	256.4912	565.2047	966.6667
Poland	2007	309.9459	234.2262	0.6488	153 295.6	64.7081	114.5833	385.1191	802.3810
Czech Republic	2016	822.0742	636.9987	0.6837	69 7699	243.2596	403.4325	891.7981	2 277.2702
Poland	2016	553.4071	370.7712	0.6767	576 099.9	119.7876	219.7162	652.5573	1 065.8505
Fitted values									
Czech Republic	2007	472.2562	383.4131	0.6281	13 5347	161.3726	248.8456	560.5548	909.4761
Poland	2007	309.9459	223.7590	0.6640	17 7134	62.8428	109.6341	373.3617	801.9741
Czech Republic	2016	822.0742	682.3778	0.6286	540 645	257.2524	409.4761	810.6297	2 112.3873
Poland	2016	533.4071	378.2716	0.6678	621 056	106.9415	212.4333	666.5040	1 009.4761

Source: Author's calculation

The compared descriptive statistics values in Table 2 are very similar in most cases. It is worth noting that the Zenga income distribution fits the mean very well. The analysis of the indexes of fitting and estimated standard errors of point estimates the Zenga income distribution to the empirical data shows that it confirms goodness of fit for a whole range of a distribution. This proves that the Zenga income distribution well describes the empirical data for the Czech Republic and Poland. Convergence to the Pareto law for high income groups and existence of only a small number of finite moments of a distributions was described in papers (Zenga, 2010a, 2010b; Zenga, Pasquazzi, Zenga, 2012). Moreover, the transparent economic interpretation of the Zenga density parameters is an additional argument to use this model to describe income distribution.

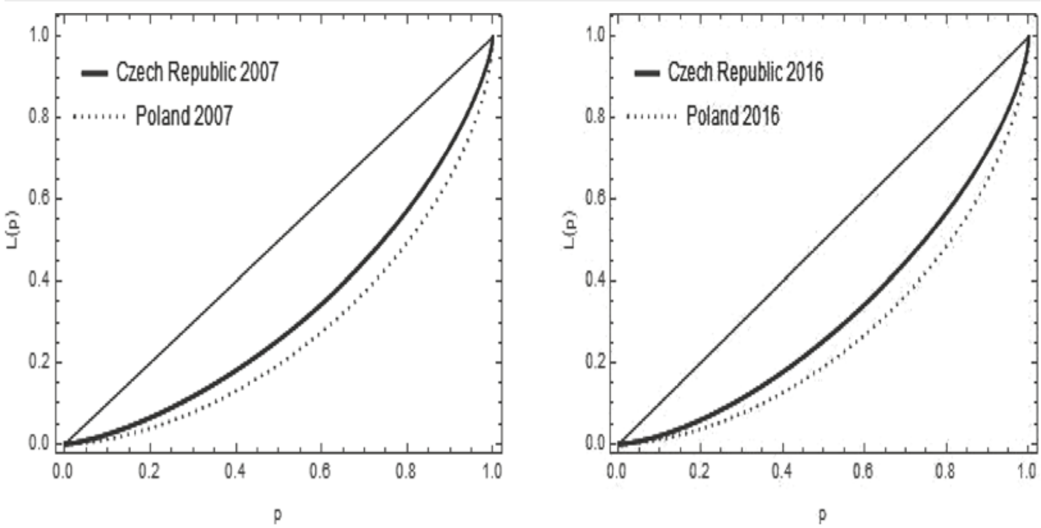
Figures 7 to 11 report graphs of the Lorenz curve $L(p)$ and the Zenga curve $I(p)$. The synthetic Gini and Zenga indexes G and Z are summarized in Table 3. These coefficients were calculated on the basis of the Zenga income distribution. The values of Gini and Zenga index estimators were obtained by means of numerical integration based on the Lorenz and Zenga curves.

Table 3 Estimation results for income distributions in the Czech Republic and Poland based on the data coming from EQLS in two period 2007 and 2016

Country	Year	Gini coefficient	Zenga coefficient
		G	Z
Czech Republic	2007	0.3591	0.7008
Poland	2007	0.4610	0.8001
Czech Republic	2016	0.3683	0.7169
Poland	2016	0.4707	0.8075

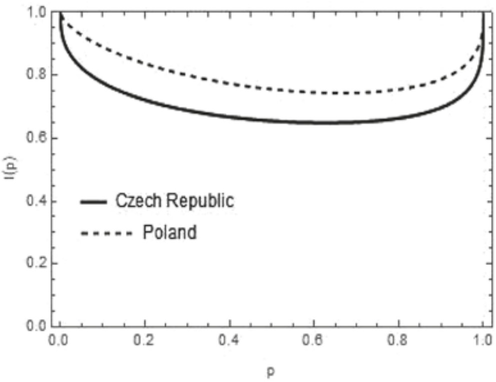
Source: Author's calculation

Figure 7 Comparison Lorenz curves $L(p)$ for the Czech Republic and Poland in two periods 2007 and 2016



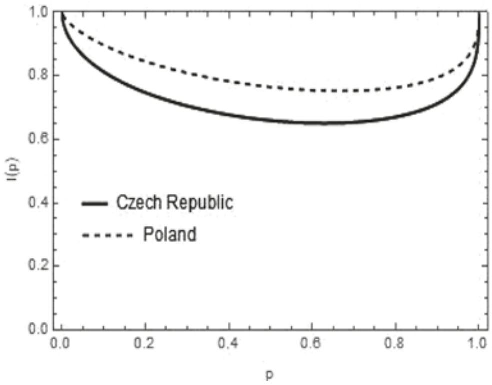
Source: Authors' calculation

Figure 8 Zenga curves *I(p)* for the Czech Republic and Poland in 2007



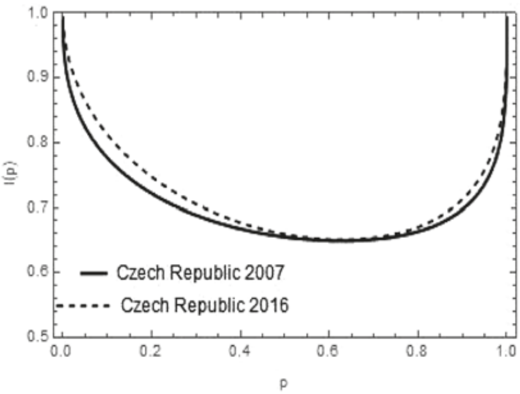
Source: Authors' calculation

Figure 9 Zenga curves *I*(*p*) for the Czech Republic and Poland in 2016



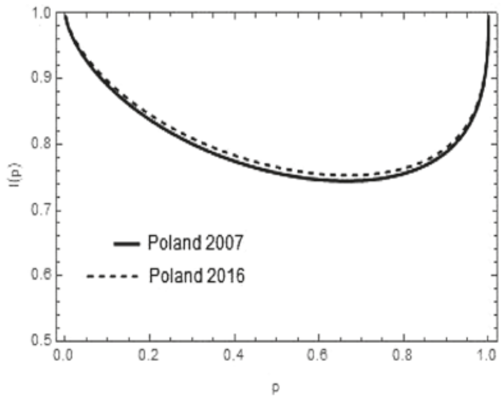
Source: Authors' calculation

Figure 10 Zenga curves *I*(*p*) for the Czech Republic in two periods 2007 and 2016



Source: Authors' calculation

Figure 11 Zenga curves *I*(*p*) for Poland in two periods 2007 and 2016



Source: Authors' calculation

Zenga's curve can be considered more explanatory and flexibile and its results possess more intuitive interpretation than Lorenz one (Polisicchio and Porro, 2009).

Comparing two populations of economic units we can also note that income inequality are slightly greater for both countries in 2016. The Gini and Zenga coefficients in two periods 2007 and 2016 for Poland do not differ significantly, we can see it in Figure 11. In Figures 8 and 9 we can discover that the Zenga inequality curves for the Czech Republic lies below the curve for Poland for the entire income range in two periods 2007 and 2016. It proves large income discrepancy between this country.

CONCLUSIONS

Income distributions in two countries (the Czech Republic and Poland) are studied for comparison. The Zenga distribution has not been used so far to analyze income distributions in the Czech Republic. The density function of the Zenga income model has a number of interesting statistical properties and thus easily adapts to various types of empirical income distributions. Zenga inome model has positive skewness, it has paretian right tail and it can be used to describe economic distributions by size.

The Zenga probability density function is characterized by a small number of parameters. The parameters of the Zenga distribution can be applied to evaluate various statistical characteristics of the distribution, in particular the point and synthetic measures of income inequality. Another worthy property of Zenga distribution is the equivalence of the parameter with the expected value, it provides the use of simple estimation methods, which is an additional advantage of this model. All these properties of the Zenga model confirm the validity of its application.

We have shown that Zenga distribution describes very well the distributions of individual incomes in both the Czech Republic and Poland. Obtained results show very interesting differences. The conducted analysis showed, using the Zenga distribution curve, how the incomes in the considered countries changed over the years. The results of the calculations concerning the level of distribution inequality, revealed differences between Czech Republic and Polish households.

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