The Usage of State Space Models in Mortality Modeling and Predictions

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Received 22.7.2022, **Accepted** (reviewed) 3.10.2022, **Published** 17.3.2023

Abstract

In demography, mortality modeling with respect to age and time dimensions is often associated with the traditionally used Lee-Carter model. The Lee-Carter model considers a constant set of parameters of age-specific mortality change for forecasts, which can lead to the problem of overcoming the biodemographic limit. The main motivation of this paper is the use of more flexible models for mortality modeling. The paper explores the use of state space models for modeling and predicting mortality in a form not typically used in demography. In this context, it is a generalized Poisson state space model with overdispersion parameters. Concerning the empirical results, a comparison is made between the predictive abilities of the Lee-Carter and the generalized Poisson state space model with overdispersion parameters. The state space Poisson model with overdispersion parameters led to better results with respect to the comparison of modeled and historical observations. However, when comparing the predictions in the cross-validation area, both models were represented with similar overall mean squared error.

Keywords	***DOI***	***JEL code***
Generalized state space models, extended Kalman filter, exponential smoothing, Lee-Carter model, mortality, prediction comparison	*https://doi.org/10.54694/stat.2022.33*	*C32, J11*

INTRODUCTION

The popularity of state space models started to rise in the late 1990s, particularly in the field of systems theory, with their utilization in the Apollo space program as a major highlight (Hutchinson, 1984). Subsequently, these models began to be used in other areas besides economic theory. State space models are based on the assumption that a time series is an output of a dynamic system that is affected by random components. State space models represent a general framework that covers a significant range of statistical models. Among the most well-known applications, one can include modeling of seasonality with a variable

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character or yield curves fitting and predicting. Yield curves are traditionally modeled with respect to the time to maturity, but state space models also allow extending this common approach to a time dimension, making it possible to predict yield curves over time into the future. Another important application is dynamic (generalized) linear models, thus generalized linear models with time-dependent parameters of the explanatory variables. State space models allow the modeling of both univariate and multivariate stationary or non-stationary time series that may contain structural changes, other irregularities, or missing observations. Finally, very popular ARIMA models (Box et al., 2008) can be also represented by state space models.

Mortality modeling plays a crucial role in solving macro and microeconomic optimization problems. From a macroeconomic perspective, it is possible to mention a need to find reliable estimates of future mortality characteristics, which significantly determine the amount of old-age pensions paid with respect to the prediction of the future demographic population structure. From a microeconomic point of view, mortality modeling is mainly dealt with by insurance companies when designing life insurance products for their portfolio.

Nowadays, one can find a large number of models that aim to find the most accurate predictions of future mortality. In this paper, the focus is held on models that describe mortality both in terms of the age structure of the population as well as in terms of the time dimension. One of the most commonly used models is the Lee-Carter model, which is characterized by a good model fit to historical data, unambiguous interpretation of its parameters from a demographic perspective (average age-specific mortality level, age-specific change in mortality interacting with a general mortality trend), and simple computational complexity when obtaining the model parameters estimates. The disadvantages of the Lee-Carter model can be found mainly in situations of very long mortality forecasts, for example for the next 50 years, where future age-specific mortality rates fall at a pace estimated from historical observations, often below the values of the biodemographic limit. According to Carnes et al. (2003), the biodemographic limit in this context is understood as a natural mortality limit, often characterized by a life expectancy that should be biologically impossible to overcome. This problem was noticed by Li, Lee, and Gerland (2013), who proposed an extension of the Lee-Carter model by rotating the age-specific set of parameters of the original Lee-Carter model.

State space models are standardly used to extend the Lee-Carter model to model general mortality trends, see Andreozzi et al. (2011) or Harvey (1990), or to incorporate the Lee-Carter model in a state space representation, see Husin et al. (2015) and Pedroza (2006). However, the application of state space models to model the overall structure of population mortality with respect to age and time dimensions has not been significantly explored so far. The main idea and the motivation of this paper resulted from the specification of the traditional Lee-Carter model, see Lee and Carter (1992). This model provides very accurate estimates of historical, already observed, mortality, however, the age-specific component of the model is considered constant over time, and, therefore, the model is not always able to capture the changing trend in age-specific mortality. For this reason, this paper aims to assess the suitability of using state space models for demographic mortality analyses compared to traditional methods of predicting mortality rates using appropriate diagnostic criteria.

1 RELATED LITERATURE

The theoretical foundations of the state space models in the traditional classical sense were originally addressed by Harvey (1990) and especially Kalman (1960). Kalman was the first who analyzed time series using state space model methodology. In terms of the literature, one can mention in particular the foundational publication by Durbin and Koopman (2012). Hyndman et al. (2008) explored the use of state space models for exponential smoothing and modeling of seasonality with a changing pattern. Petris et al. (2009) explored the Bayesian approach to estimate the parameters of state space models. The prediction of state space models from the Bayesian perspective was also addressed by Harrison and Stevens (1976).

A basic description of state space models as well as an overview of the specific applications can be found in Slavík (2005). The use of state space models in demography was addressed also by Matějka (2017).

There is currently no universal notation of state space models. However, there are two major commonly used notations across statistical and econometric applications. The first, also used in this paper, originally based on the approach to state space models as defined by Harvey (1990), was later used as the most commonly used notation in econometrics by Durbin and Koopman (2012). The second notation is based on the original study of state space models by Harrison and Stevens (1976) and, due to additional use by West and Harrison (2006), is often used primarily in systems analysis and control processes. However, this notation can be encountered in the literature primarily concerning Bayesian approaches to state space models.

The Lee-Carter model, introduced by Lee and Carter in 1992, has become one of the most widely used models in demographic analyses of mortality prediction, affecting both age and time dimensions. Its widespread use implied the need to refine the original model to consider specific case studies. In this context, the extension of the original Lee-Carter model with an additional set of parameters covering not explained age-specific mortality of the original Lee-Carter model (log-linear model, see D'Amato et al. 2011), or the incorporation of cohort dependence of mortality (Age-Period-Cohort, Lee-Carter, see Renshaw and Haberman, 2006) can be mentioned. Another extension of the Lee-Carter model was done by Li, Lee, and Gerland (2013) who introduced time-rotating parameters for age-specific mortality changes.

In demography, state space models are mainly used in the context of population predictions, see Tavecchia et al. (2009). The statistical representation of mortality patterns using a state space model with a Markov process to define state variables was investigated by Fung et al. (2017). Several studies have examined the benefits of combining the state space model approach and the Lee-Carter representation. For example, an extension of the Lee-Carter model in terms of simultaneous estimation and prediction of a time-dependent set of parameters was considered by Reichmuth and Sarferaz (2008), who focused on predicting mortality in the US until 2050. Zakiyatussariroh et al. (2014) focused on comparing the estimates of the Lee-Carter model and the corresponding representation using state space models to model mortality in Malaysia. Husin et al. (2015) subsequently extended this study by comparing the predictions of the two mentioned models concerning short-term and long-term predictions (compared with the original Lee-Carter model). Several studies have also focused on linking the Lee-Carter model and a state space model, where the evolution of specific mortality rates is defined by an observation equation, see Lazar and Denuit (2009) and Pedroza (2006), who additionally used the Bayesian approach to improve prediction characteristics of the general mortality trend. The states equation in this case defines the evolution of the general mortality trend. Fung et al. (2015) considered extending the Lee-Carter model using a Bayesian state space model approach to improve annuity price estimates. State space models are not very frequently used in demography, however, one can mention a study of Abd Nasir et al. (2021) who investigated the prediction of the evolution of under-5 child mortality using a state space local linear trend model. The same model was used by Khedhiri (2021) to predict the evolution of Covid-19 infectivity in the Arab States. Andersson and Lindholm (2021) extended the use of the random walk approach with the Lexis diagram to model and predict mortality using state space models.

The estimation of state space model parameters is a very computationally demanding task from a practical point of view, therefore it is possible to find several libraries implemented in the statistical programming environment R (Team R, 2021) that address this issue, see Petris and Petrone (2011) or Tussel (2011). These are, for example, KFAS (Helske, 2017), dlm (Petris, 2010), dse (Gilbert, 2009), sspir (Dethlefsen et al., 2009). The statistical computing interface EViews (Van Den Bossche, 2011) or the SsfPack libraries in the Oxmetrics computing environment (Mendelssohn, 2011) can also be used to solve state space models.

2 DATASET AND STATISTICAL METHOD

The empirical study in this paper was carried out using the publicly available data from HMD (Human Mortality Database),[3] where the number of deaths (referring to the third main sets of Lexis diagram) and the exposure of risks of the population for males in the Czech Republic are available. The considered models are constructed for the historical years in the training area from 1950 to 2012, while validation data for the period 2013 to 2019 are used to verify the ability of each model to predict accurately future mortality rates. In terms of the age dimension, five-year age categories from 0, 1–4 to 95+ years (21 categories) are considered. Thus, the training dataset has a dimension of (21×63) and the validation dataset has a dimension of (21×7).

The data is applied to the Lee-Carter model and the multivariate Poisson state space model, which will be outlined in the following chapters by moving from the Gaussian state space model to the generalized state space model.

2.1 Gaussian state space model

Linear Gaussian state space models consist of two sets of equations, these are the observation equation and the state equation. The notation of state space models, including state matrices and vectors, is consistent with the definition of state space models used by Durbin and Koopman (2012) and Helske (2017). The first set of p equations explains the behavior of the observed data using observable or unobservable (latent or state) variables and thus can be written as:

$$y_t = Z_t\alpha_t + \varepsilon_t \,, \tag{1}$$

where $t = 1, 2, ..., n$ is the time index, y_t is a vector of a multivariate observation time series (*observation vector*) of a dimension $(p \times 1)$, $\varepsilon_t \sim N_p(0, H_t)$ is a p-dimensional vector of random terms. The vector of observed or unobserved latent variables (*state vector*) α_t of a dimension $(m \times 1)$ explains y_t by a state matrix Z_t of a dimension $(p \times m)$.

The second set of m equations describes the evolution of latent variables over time, again influenced by random terms:

$$\alpha_{t+1} = T_t\alpha_t + R_t\eta_t \,, \tag{2}$$

where $\eta_t \sim N_k(0, Q_t)$ is a k-dimensional vector of random terms affecting the latent variables α_{t+1}. System matrices T_t of a dimension $(m \times m)$ and R_t of a dimension $(m \times r)$ define the relationships within the state and observation equations. Covariance matrices H_t and Q_t determine the covariance structure of the random terms for each equation, see Durbin and Koopman (2012). The covariance matrices H_t and Q_t can be in some cases presumed time-invariant, and thus one can preferably consider only a matrix H of a dimension $(p \times p)$ and a matrix Q of a dimension $(r \times r)$. The random terms ε_t and η_t are assumed to be uncorrelated with each other for the entire time axis t. They are further assumed to be uncorrelated with the initial state vector α_1.

2.2 Kalman filter and smoother

The main procedures used to estimate the parameters of classical state space models are the Kalman filter and the Kalman smoother, see Durbin and Koopman (2012) and Tusell (2011). The Kalman filter refers to a recursive procedure leading to filtered estimates of unknown state variables α_t for observations of (multivariate) time series y_t, for $t = 1, 2, ..., n$. Kalman filter is a procedure that uses filtering as a process where one-step ahead estimates of state variables are made more precise as new time series observations y_t are taken into consideration.

[3] *<www.mortality.org>*.

Kalman smoother is a backward recursive algorithm for $t = n, n - 1, ..., 1$ that uses the estimated state variables of the Kalman filter to obtain smoothed estimates of the state variables based on all observations, see Koopman and Durbin (2001).

As already mentioned, the Kalman filter and the smoother are recursive procedures. In general, a distinction can be made between recursive and non-recursive approaches. In the case of non-recursive filters, the basic idea is to gradually incorporate information from the observed (multivariate) time series to obtain filtered estimates. However, none of the sequentially obtained filtered estimates are taken into account in the calculation of subsequent estimates. In contrast to this approach, recursive filtering methods aim at obtaining filtered estimates of the model parameters at time n, whereby the estimate is constructed based on the already obtained filtered estimates corresponding to time $n - 1$. Both approaches can be iterative, where the filtered estimate at time n is used multiple times (with respect to the same observations) in order to obtain more accurate filtered estimates.

Koopman and Durbin (2001) outline a detailed derivation of the extended Kalman filter and exponential smoothing.

2.3 Multivariate State Space Poisson model with overdispersion parameters

Gaussian state space models can be extended to consider probability distributions of an exponential family of distributions, thus relaxing the assumption of a normal distribution or the linear dependence in Formulas (1) and (2). The observation equation is then expressed in terms of the probability density of the random variables of the observed time series:

$$p(y_t|\theta_t) = p(y_t|Z_t \alpha_t) \, , \tag{3}$$

and the state vector is defined as:

$$\alpha_{t+1} = T_t \alpha_t + R_t \eta_t \, , \tag{4}$$

where $p(y_t|\theta_t)$ is the probability density of the random variables of the observed time series and $\theta_t = Z_t \alpha_t$ is referred to as the signal. Thus, the signal is a linear predictor that explains the expected value $E(y_t) = Z_t \alpha_t$ of the random variable y_t using a link function $g(u_t) = \theta_t$. The probability density $p(y_t|\theta_t)$ may follow a non-normal distribution or be non-linear, respectively both situations may occur. If the density $p(y_t|\theta_t)$ follows the assumption of a normal distribution and the signal θ_t is a linear function of y_t, then the model represented by Formulas (3) and (4) transfers to the Gaussian state space model defined by Formulas (1) and (2).

For the use of Formulas (3) and (4), it is possible to use the standard procedures that are well known in the field of generalized linear models. For the purpose of this paper, we mention the Poisson distribution with the expected value λ_t, exposure u_t and the logarithmic link function $\theta_t = \log(\lambda_t)$, thus $E(y_t|\theta_t) = D(y_t|\theta_t) = u_t e^{\theta_t}$.

The core model of the presented paper is a multivariate Poisson linear state space model with overdispersion character (hereafter referred to as the PSSO model). Its theoretical background is the exponential state space model defined by the Formulas (3) and (4), and therefore the PSSO model is defined as follows:

$$\begin{aligned} & p(y_t|\theta_t) = \mathrm{Po}_{20}(u_t e^{\theta_t}), \\ & \theta_t = u_t + \varepsilon_t, \ \varepsilon_t \sim N_{20}(0, Q_\varepsilon), \\ & \mu_{t+1} = \mu_t + \nu_t + \xi_t, \ \xi_t \sim N_{20}(0, Q_\xi), \\ & \nu_{t+1} = \nu_t \, , \end{aligned} \tag{5}$$

where μ_t is a random walk process, ν_t is a constant slope, and ξ_t is the component capturing the additional variance in the time series (overdispersion). No constraints are being considered with respect to the covariance matrices Q_ε and Q_ξ, thus general matrices are assumed. Dimension of vectors y_t, θ_t, μ_t, ν_t, ε_t a η_t is (21 × 1). Dimension of matrices Q_ε and Q_ξ is (21 × 21) for $t = 1, 2, ..., 63$.

The construction of the model (5) was performed using the KFAS library (Helske, 2017) in the statistical software R (Team R, 2021), using standard functions for the state space model specification.

2.4 Extended Kalman filter and exponential smoothing

In the case of generalized state space models, it is necessary to use the extended Kalman filter that first linearizes the generalized state space model using the Laplace approximation and then applies the standard Kalman filter and smoother. The estimation of the unknown parameters of the model (state variables, covariance matrices, or only their parts) is performed in the traditional way using the maximum likelihood method with respect to the considered probability distribution of the exponential family. Numerical optimization is then performed by the BFGS method, where this abbreviation is derived from the initial letters of the names of the independently published authors Broyden (1970), Fletcher (1970), Goldfarb (1970), and Shanno (1970), or possibly by the Nelder-Mead method, according to Nelder and Mead (1965).

A detailed derivation of the extended Kalman filter and exponential smoothing can be found in Koopman and Durbin (2001).

2.5 Standard Lee-Carter model

A very useful, long-standing, and still popular approach for modeling and predicting mortality rates is the Lee-Carter log-bilinear model (hereafter referred to as the LC model). The method proposed by Lee and Carter (1992) has become a fundamental statistical method used in demographic analyses. The LC model aims to model and predict age- and time-specific mortality rates $m_{x,t}$, which are defined by the model as:

$$\log(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}, \tag{6}$$

where t represents a time, x is an age (respectively an age category) for $t = 1, 2, ..., n$ and $x = 1, 2, ..., r$. A matrix $m_{x,t} = D_{x,t} / E_{x,t}$ of a dimension $(r \times n)$ represents age- and time-specific mortality rates. According to Šimpach and Arltová (2016), it is presumed that random terms $\varepsilon_{x,t}$ follow a normal distribution with zero mean, constant variance, and meet the assumptions of the white noise process. The number of deaths at age x at time t is denoted as $D_{x,t}$ and the mean population at age x and time t is denoted as $E_{x,t}$.

The first set of parameters of the model (6) contains r parameters α_x and it defines an age-specific general mortality profile, hence it is a vector of a dimension $(r \times 1)$. The second set again contains r parameters β_x and represents the deviations of the mortality rate from α_x as a result of an interaction with the general mortality trend, which is determined by the third set of parameters κ_t. Thus, the third set contains t parameters, respectively it is a vector of a dimension $(1 \times t)$.

The estimation of the unknown parameter sets of the Lee-Carter model by the maximum likelihood method is based on the assumption that the number of deaths $D_{x,t}$ at age x and at time t is a random variable that follows a Poisson distribution, see Brillinger (1986) or Wilmoth (1993). The expected value of the random variable $D_{x,t}$ as well as the constrains applied due to model (6) parameters unambiguity can be found in the respective literature, see Lee and Carter (1992), Wilmoth (1993), Girosi and King (2007) or Richards and Currie (2009). Other estimation methods, such as singular value decomposition or weighted least squares, can be also alternatively used, see Andreozzi et al. (2011) or Koissi and Shapiro (2008).

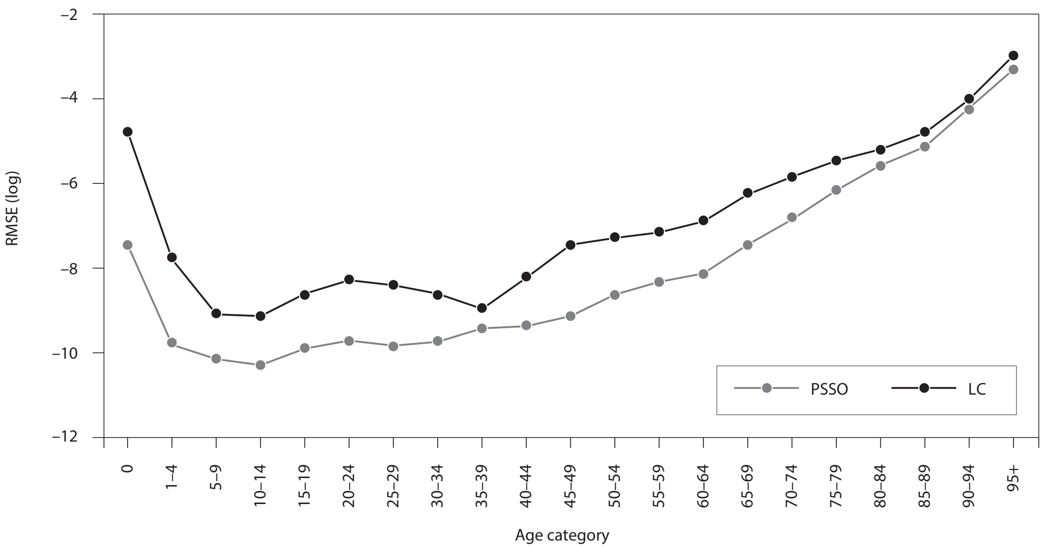
[2] *<www.mortality.org>*.

3 EMPIRICAL FINDINGS

The main purpose of mortality models is their ability to accurately predict future population mortality. This ability may be affected by the presence of a systematic nature of mortality, which can be revealed during a residual diagnosis, but also when comparing modeled and historical observations.

Figure 1 shows the Root-Mean-Square error (RMSE) of the Lee-Carter (LC) and Poisson state space model with overdispersion parameters (PSSO) fitted values compared to the historical data. From the values shown, it is apparent that the PSSO model achieves a more accurate fit of the model mortality rates to historical observations when compared to the LC model for all age categories.

Figure 1 Age pattern of the RMSE in the logarithmic scale of the PSSO and LC models of historical training data 1950–2012



Source: Own construction

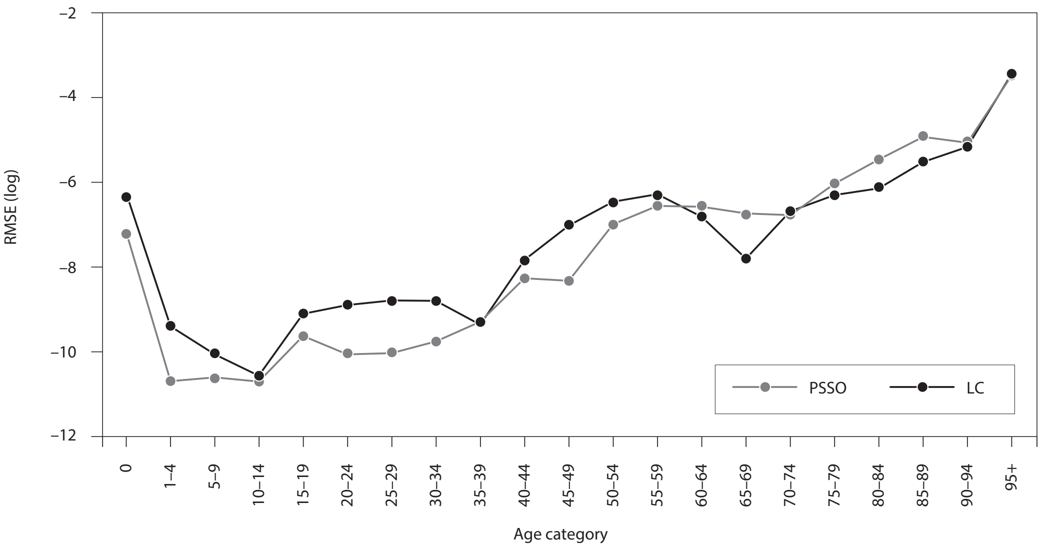
The age pattern of RMSE on a logarithmic scale is illustrated in Figure 1. The first age category of 0 years old is characterized by a higher RMSE value for both models. These deviations subsequently decrease until the age category 10–14 years. For the LC model (darker line), a saddle of RMSE values can be observed which ends with the age category 35–39 years. RMSE values subsequently increase with increasing age categories. According to the age pattern of the RMSE of the PSSO model (lighter line), it can be seen that this model provides lower RMSE values for all age categories and is, therefore, able to provide more accurate modeled values for specific mortality rates when compared to the LC model. The total RMSE of the LC model is 0.10147, respectively 0.06466 in the case of the PSSO model.

As already mentioned, the main motivation for constructing mortality models is to obtain the most accurate mortality predictions. For this purpose, the PSSO and LC models were estimated over the 1950–2012 historical training period and the validation period 2013–2019 was used to compare the predictions of these models with known observations.

Figure 2 shows the age pattern of the RMSE values of the PSSO (lighter line) and the LC model (darker line) on a logarithmic scale for the validation area.

The total RMSE of the LC model is 0.055. In the case of the PSSO model, the average RMSE is 0.057. Figure 2 shows that the evolution of RMSE can be divided into three areas for each age category.

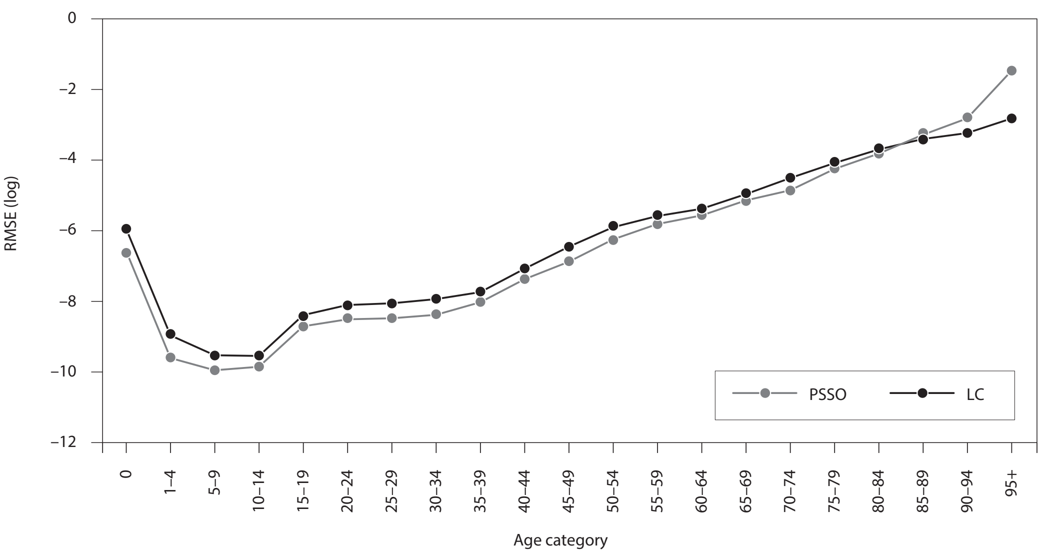
Figure 2 Age pattern of RMSE in logarithmic scale of PSSO and LC models for validation area 2013–2019



Source: Own construction

For the first area, age categories 0 to 55–59, the PSSO model provides a lower RMSE (except for age category 35–39). In the second area from age categories 60–64 to 90–94, the RMSEs are lower for the LC model (except for age categories 65–59 and 70–74). For the last area, the highest ages of 95+ years, the RMSE values are similar but lower for the PSSO model, such conclusion is valid also for the age category

Figure 3 Age pattern of RMSE in logarithmic scale of PSSO and LC models confidence prediction intervals for validation area 2013–2019



Source: Own construction

90–94 with the opposite effect. It can be concluded, that both models are characteristic with a similar total RMSE, however, the PSSO model provides a better fit of predicted values to known observations in two-thirds cases (14 out of 21 time series). Specifically, these time series are 0, 1–4, 5–9, 10–14, 15–19, 20–24, 25–29, 30–34, 40–44, 45–49, 50–54, 55–59, 70–74, and 95+ years, see Figure 2.

Figure 3 shows the age pattern of the RMSE values of the PSSO (lighter line) and the LC model (darker line) on a logarithmic scale for the validation area when comparing the respective confidence intervals of the predictions and the observed mortality rates in the validation period.

The total RMSE of the LC model confidence prediction intervals to the validated mortality rates is 0.206. In the case of the PSSO model, the average RMSE is 0.387. Despite the fact that the total RMSE is lower for LC model, the age pattern of RMSE in Figure 2 shows that the RMSE of mortality rates and confidence prediction intervals of both models is smaller for PSSO model for all age categories expect for the older ages (85–89, 90–94 and 95+) where the accuracy of both models is limited due to less data availability from a demographic perspective.

CONCLUSION

The aim of the paper was to find a more flexible model that would take into account age-specific changes in mortality over time. The Lee-Carter model was chosen as a benchmark to such a model. The Lee-Carter model considers the invariant nature of age-specific mortality change due to the second set of parameters, which interact with the general mortality trend to determine the future predictions of mortality.

In the empirical part of this paper, the generalized state space and Lee-Carter models are applied to Czech mortality data from 1950 to 2012 (training part of the data). Information for the period from 2013 to 2019 (validation part of the data) is then used to assess the predictive ability of both models. The generalized state space model, as specified in this paper, is not standardly used in the field of demographic analysis, and thus the introduction of this model alone reveals a wide range of potential applications of state space models for mortality or other demographic characteristics modeling.

Summing up the results, it can be concluded that this study has shown that both PSSO and LC models show a similar RMSE pattern with respect to the model fit to data. Although the age pattern of RMSE is similar, the PSSO model provides lower RMSE and, therefore, more accurate model values of specific mortality rates for all age categories in the training data area.

When comparing the deviations of the predicted values from the known observations, it can be concluded that the total RMSE deviation is similar for both PSSO and LC models. In terms of the age patterns of RMSE for each age category, three areas can be observed, corresponding to lower RMSE values for the PSSO model, lower RMSE values for the LC model, and, finally, the last area of similar RMSE values for both models. When assessing the RMSE values for all age categories, it can be observed that the PSSO model shows a better fit between predictions and observations in the training area in two-thirds of the cases when compared to the LC model.

Both models were also compared from their prediction intervals accuracy perspective. The respective age specific RMSE revealed that the usage of PSSO model resulted in lower RMSE for all age categories except the oldest three categories (85–89, 90–94 and 95+) which can be summarized as lower RMSE in 85 % of cases for the PSSO model.

The computational complexity of estimating the PSSO model is significantly higher when compared to the time required to obtain the LC model estimates. The computational complexity was reduced by estimating the model first without simulations. These parameter estimates were used as initial values for the Importance sampling method with the BFGS optimization procedure, which provided model estimates in less time when compared to the Nelder-Mead method. The resulting estimates obtained from these two methods hardly differed. However, the computation of the estimation of the state variables of the PSSO model using the above approach takes still several hours, whereas the estimation of the parameters of the LC model takes only a few seconds. Contrary to LC model parameters where their interpretation is meaningful from a demographic perspective, the interpretation of PSSO state parameters is rather cryptic and hence difficult to interpret.

The use of the PSSO model, in this specific use case, resulted in similar overall RMSE values in comparison to LC model. Despite the fact that age patterns of RMSE were more precise for two-thirds of age categories (respectively in 85% cases when focused on confidence intervals), it cannot be concluded that PSSO model would be recommended for mortality predictions. It worth to be mentioned that the conclusion above was done by using both models on Czech male mortality data, hence further evaluation of more data sets is recommended in order to obtain more conclusive results.

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