Statistical Inference Based on L-Moments

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Abstract

To overcome drawbacks of central moments and comoment matrices usually used to characterize univariate and multivariate distributions, respectively, their generalization, termed L-moments, has been proposed. L-moments of all orders are defined for any random variable or vector with finite mean. L-moments have been widely employed in the past 20 years in statistical inference. The aim of the paper is to present the review of the theory of L-moments and to illustrate their application in parameter estimating and hypothesis testing. The problem of estimating the three-parameter generalized Pareto distribution's (GPD) parameters that is generally used in modelling extreme events is considered. A small simulation study is performed to show the superiority of the L-moment method in some cases. Because nowadays L-moments are often employed in estimating extreme events by regional approaches, the focus is on the key assumption of index-flood based regional frequency analysis (RFA), that is homogeneity testing. The benefits of the nonparametric L-moment homogeneity test are implemented on extreme meteorological events observed in the Czech Republic.[2]

Keywords	JEL code
L-moment, parameter estimation, generalized Pareto distribution, homogeneity testing, precipitation extreme events, Czech Republic	*C02, C12, C13, C14, C15*

INTRODUCTION

Moments, such as mean, variance, skewness and kurtosis, are traditionally used to describe features of a univariate distribution. Hosking (1990) introduced an alternative approach using L-moments, which are defined as certain linear combinations of order statistics. The main L-moments' advantage, in comparison to conventional moments, is their existence of all orders under only a finite mean assumption. When describing a multivariate distribution, the situation is very similar. The mean vector and covariance, coskewness and cokurtosis matrices with elements the covariance, coskewness and cokurtosis are the characteristics usually used to summarize features of a multivariate distribution. However, central comoments (covariance, coskewness, cokurtosis, etc.) are defined under finiteness of central moments of lower orders. To avoid this drawback, Serfling and Xiao (2007) proposed multivariate L-moments with elements the L-comoments as analogues to central comoments, without giving assumptions to finiteness of second and higher central moments.

L-moments, being measures of shape of a probability distribution, may be used for summarizing data drawn from both univariate and multivariate probability distributions. Besides description statistics,

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L-moments play an important role also in inferential statistics. In the past 25 years the method of L-moments has been used as a convenient alternative to the traditional estimation method of moments and maximum likelihood method, mainly in hydrology, climatology and meteorology (e.g., Kyselý and Picek, 2007), but also in economics and socioeconomics (e.g., Bílková, 2014). The L-moments based estimates are obtained in a similar way as in the moment method, which means the population L-moments are equated to their corresponding sample quantities. Hosking (1990) gives parameter estimators of some common univariate distributions and highlights L-moments, because they sometime provide better estimates than the maximum likelihood method (particularly for small samples and heavy-tailed distributions). Several other studies have shown that the L-moment method in some cases outperforms also other estimation methods, including the well-known method of moments and relatively new methods of TL- and LQ-moments when estimating their parameters or high quantiles (Hosking, Wallis and Wood, 1985; Hosking and Wallis, 1987; Martins and Stedinger, 2000; Delicado and Goria, 2008; Šimková and Picek, 2016), as well. Moreover, L-moments based estimates are more tractable than maximum likelihood estimates. Besides parameter estimating, L-moments are also employed in hypothesis testing, particularly in RFA which yields reliable estimates of high quantiles of extreme events using data from sites, which have similar probability distributions. A univariate approach based on L-moments introduced by Hosking and Wallis (1997) has been routinely used in areas such as hydrology, climatology and meteorology, among others (Chen et al., 2006; Kyselý, Picek and Huth, 2006; Kyselý and Picek, 2007; Viglione, Laio and Claps, 2007; Noto and La Loggia, 2009; Kyselý, Gaál and Picek, 2011). Attention to multivariate RFA has been devoted recently in works of Chebana and Ouarda (2007), and Chebana and Ouarda (2009), in which the main steps of univariate index-flood based RFA of Hosking and Wallis (1997) were generalized using multivariate L-moments, copulas and quantile curves. Now multivariate RFA based on L-moments becomes popular in practice, because it improves analysis of the studied phenomenon by considering more available information (Chebana et al., 2009; Ben Aissia et al., 2015; Requena, Chebana and Mediero, 2016).

The paper gives a brief review of the theory of L-moments and their selected applications and uses already known methods to illustrate their use in specific examples in practice. First, the usefulness of L-moments is shown in the problem of estimating the GPD parameters often used in modelling extreme events. Although various techniques, such as the moments, L-moments or maximum likelihood methods, have been proposed in the literature for estimating parameters of a probability distribution, some of them are more accurate for data of certain properties as it has been already shown in several comparison studies (Hosking, Wallis and Wood, 1985; Hosking and Wallis, 1987; Martins and Stedinger, 2000; Delicado and Goria, 2008). Hence, a small simulation study is performed to compare several estimation methods and to show the superiority of the method based on L-moments for estimating GPD parameters in some cases. However, nowadays L-moments are mainly used in RFA to reliably estimate high quantiles of extreme events. The second illustration uses L-moments in hypothesis testing. Several papers have already dealt with both univariate and multivariate RFA based on L-moments of extreme precipitation events in the Czech Republic (Kyselý, Picek and Huth, 2006; Kyselý and Picek, 2007; Kyselý, Gaál and Picek, 2011; Šimková, Picek and Kyselý, in preparation). All these studies employed for homogeneity checking the parametric Hosking and Wallis (1997) or generalized Chebana and Ouarda (2007) L-moment homogeneity tests, which preceded model's parameters estimation and also relatively labouring selection of the best copula in the bivariate case. The nonparametric procedure is more powerful and easier to implement than the parametric one, because it does not require estimation of model's parameters nor specification of the copula in the multivariate case, and, hence, the homogeneity testing becomes simpler and quicker. Here, the benefits of the nonparametric homogeneity testing based on L-moments are implemented on bivariate extreme meteorological events observed in the Czech Republic. We will investigate whether the regions' homogeneity will be confirmed also by the nonparametric test, but in much easier way.

Because the nonparametric test of Masselot, Chebana and Ouarda (2016) has been introduced very recently, this is one of the first attempts of implementing the nonparametric procedure on the real-world data.

The paper is organized as follows: In the first section, the theory of both univariate and multivariate L-moments and their use in statistical inference are briefly reviewed. The use of L-moments in specific tasks, particularly in estimating the GPD parameters and nonparametric checking of regions' homogeneity formed in the area of the Czech Republic, and results obtained are presented in Section 2. The paper closes with summary section.

1 METHODOLOGY

1.1 Univariate L-moments

1.1.1 Population univariate L-moments

A population L-moment is defined to be a certain linear combination of order statistics (the letter L just emphasizes that the L-moment is a linear combination) which exists for any random variable with finite mean. Hosking (1990) defined the population L-moment of the *r*th order as a linear combination of the expectations of the order statistics $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ of a random sample of size *n* drawn from a univariate distribution of a random variable *X* with cumulative distribution function *F:*

$$\lambda_r = \frac{1}{r}\sum(-1)^k \binom{r-1}{k} EX_{r-k:r}, r = 1,2,\ldots \tag{1}$$

When comparing L-moments to conventional moments, L-moments have some merits, including their existence, uniqueness and robustness (because they are linear combinations). The Formula (1) may be rewritten to the form, which is useful particularly for computation of L-moments of a given probability distribution,

$$\lambda_r = \int_0^1 x(F)P_{r-1}^*(F(x))dF, r = 1,2,\ldots, \tag{2}$$

where:

$$P_r^*(t) = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\binom{r+i}{i}t^i$$

is the *r*th shifted Legendre polynomial and *x(F)* is quantile function of a variable *X*. The first L-moment is just the mean of a random variable *X* and the second L-moment is equal to one-half of the Gini's mean difference statistic. Serfling and Xiao (2007) also present the expression of the second and higher order L-moments in the covariance representation as:

$$\lambda_r = \mathrm{cov}(X, P_{r-1}^*(F(X))), r \geq 2. \tag{3}$$

It is desirable to define dimensionless versions of higher L-moments, termed L-moment ratios, as:

$$\tau_r = \frac{\lambda_r}{\lambda_2}, r \geq 3.$$

Analogy of the coefficient of variation may be also defined in terms of L-moments as the ratio of the second L-moment λ_2 to the first L-moment λ_1

$$\tau = \frac{\lambda_2}{\lambda_1}.$$

The first two L-moments λ_1 and λ_2, termed L-location and L-scale, being measures of location and scale, and the third and fourth L-moment ratios τ_3 and τ_4, termed L-skewness and L-kurtosis, being measures of skewness and kurtosis, may be used for summarizing a univariate distribution. See Table 1 for the first four L-moments of some selected common univariate distributions which may be simply derived using the Formula (2).

Table 1 L-moments of several selected univariate distributions

Distribution	Quantile function	L-moments
Uniform	$x(F) = \alpha + (\beta - \alpha)F$	$\lambda_1 = \frac{1}{2}(\alpha + \beta), \lambda_2 = \frac{1}{6}(\beta - \alpha)$, $\lambda_3 = 0, \lambda_4 = 0$
Exponential	$x(F) = \xi - \alpha \log(1 - F)$	$\lambda_1 = \xi + \alpha, \lambda_2 = \frac{1}{2}\alpha, \lambda_3 = \frac{1}{6}\alpha$, $\lambda_4 = \frac{1}{12}\alpha$
Normal	no explicit form, approximation used $x(F) \doteq \mu + 5.063\sigma[F^{0.135} - (1 - F)^{0.135}]$	$\lambda_1 = \mu, \lambda_2 = \pi^{-\frac{1}{2}}\sigma, \lambda_3 = 0$, $\lambda_4 = 0.0702\sigma$
Logistic	$x(F) = \xi + \alpha \log\left(\frac{1 - F}{F}\right)$	$\lambda_1 = \xi, \lambda_2 = \alpha, \lambda_3 = 0, \lambda_4 = \frac{1}{6}\alpha$
Generalized Pareto	$x(F) = \xi + \frac{\alpha}{k}[1 - (1 - F)^k]$	$\lambda_1 = \xi + \frac{\alpha}{k + 1}$, $\lambda_2 = \frac{\alpha}{(k + 1)(k + 2)}$, $\lambda_3 = \frac{\alpha(1 - k)}{(k + 1)(k + 2)(k + 3)}$, $\lambda_4 = \frac{\alpha(k - 1)(k - 2)}{(k + 1)(k + 2)(k + 3)(k + 4)}$
Generalized extreme-value	$x(F) = \xi + \frac{\alpha}{k}[1 - (-\log F)^k]$	$\lambda_1 = \xi + \frac{\alpha}{k} - \alpha\Gamma(k)$, $\lambda_2 = \alpha\Gamma(k)(1 - 2^{-k})$, $\lambda_3 = \alpha\Gamma(k)(2 \cdot 3^{-k} + 3 \cdot 2^{-k} - 1)$ $\lambda_4 = \alpha\Gamma(k)(-5 \cdot 4^{-k} + 10 \cdot 3^{-k} - 6 \cdot 2^{-k} + 1)$

Source: Hosking (1990)

Although L-moments do not exist for distributions which have no finite mean (e.g., this happens for a Cauchy distribution, or generalized Pareto and generalized extreme-value distributions for certain values of the shape parameter), generalizations of L-moments, termed trimmed L-moments (abbreviated TL-moments) and LQ-moments, have been proposed. They always exist. See Elamir and Seheult (2003) and Mudholkar and Hutson (1998) for their definitions and properties.

1.1.2 Sample univariate L-moments

Because L-moments are defined for a probability distribution, they must be in practice estimated from an observed random sample drawn from an unknown probability distribution. The rth sample L-moment, being an unbiased estimator of the population L-moment λ_r, defined Hosking (1990) as a linear combination of the order sample $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ of size n

$$l_r = \binom{n}{r}^{-1} \sum_{1\leq i_1} \sum_{<i_2<\ldots} \cdots \sum_{<i_r\leq n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}, r = 1,2,\ldots,n.$$

The first sample L-moment termed sample L-location, is equal to the sample mean, while the second sample L-moment is called sample L-scale.

Naturally, the L-moment coefficient of variation τ and L-moment ratios τ_r are estimated by the sample L-moment coefficient of variation and sample L-moment ratios given by:

$$t = \frac{l_2}{l_1}, t_r = \frac{l_r}{l_2}, r \geq 3. \tag{4}$$

Observed data may be alternatively summarized and described by the sample L-location l_1, L-scale l_2, L-skewness t_3 and L-kurtosis t_4.

1.1.3 Method of L-moments

Usually, the method of maximum likelihood and method of moments are used for estimation of parameters of a probability distribution. Following the same idea as in the case of method of moments, L-moments provide parameter estimates. Let X be a random variable with a probability density function $f(x; \theta_1, \ldots, \theta_k)$, where $\theta_1, \ldots, \theta_k$ are k unknown parameters. The unknown parameters are estimated by solving the system of equation which arise from matching the first k population L-moments with corresponding sample counterparts, i.e.,

$$\lambda_i = l_i, i = 1,\ldots,k. \tag{5}$$

Hosking (1990), and Hosking and Wallis (1997) give parameter estimates of selected common univariate probability distributions derived by the L-moment method. Parameter estimates of some univariate probability distributions are shown in Table 2.

Table 2 Parameter estimation of several selected univariate distributions

Distribution	Parameter estimation
Uniform	$\hat{\alpha} = l_1 - 3l_2, \hat{\beta} = 2l_1 - \hat{\alpha}$
Exponential	$\hat{\alpha} = 2l_2, \hat{\xi} = l_1 - \hat{\alpha}$
Normal	$\hat{\mu} = l_1, \hat{\sigma} = \sqrt{\pi}\, l_2$
Logistic	$\hat{\xi} = l_1, \hat{\alpha} = l_2$
Generalized Pareto	$\hat{k} = \frac{1 - 3t_3}{1 + t_3}, \hat{\sigma} = l_2(\hat{k}+1)(\hat{k}+2), \hat{\xi} = l_1 - \frac{\hat{\sigma}}{\hat{k}+1}$
Generalized extreme-value	$\hat{k} \approx 7.859z + 2.9554z^2$, where $z = \frac{2}{3+t_3} - \log_3 2, \hat{\sigma} = \frac{l_2}{\Gamma(\hat{k})(1-2^{-\hat{k}})}, \hat{\xi} = l_1 - \frac{\hat{\sigma}}{\hat{k}} + \hat{\sigma}\Gamma(\hat{k})$

Source: Hosking (1990)

1.2 Multivariate L-moments

1.2.1 Population L-comoments

Serfling and Xiao (2007) defined L-comoments, which describe a multivariate distribution only under finite mean assumptions, analogously to the forms of central comoments and univariate L-moments in the covariance representation given by the Formula (3). Hence, it is worth remembering central comoments.

Let's have a bivariate random vector (X_1, X_2) with cumulative distribution function F, marginal distribution functions F_1, F_2, finite means μ_1, μ_2 and central moments $\mu_k^{(1)}, \mu_k^{(2)}, k \geq 2$. The rth central comoment of variable X_1 with respect to variable X_2 is defined as:

$$\xi_{r[12]} = \mathrm{cov}(X_1, (X_2 - \mu_1^{(2)})^{r-1}), r \geq 2.$$

The second, third and fourth central comoments $\xi_{2[12]}, \xi_{3[12]}, \xi_{4[12]}$ are covariance, coskewness and cokurtosis, respectively. Dimensionless versions of central comoments are given by:

$$\psi_{r[12]} = \frac{\xi_{r[12]}}{\sqrt{\mu_2^{(1)} (\mu_2^{(2)})^{r-1}}}, r \geq 2.$$

The second, third and fourth central rescaled comoments $\psi_{2[12]}, \psi_{3[12]}, \psi_{4[12]}$, are called correlation, coskewness and cokurtosis coefficients, respectively.

Let's have a bivariate random vector (X_1, X_2) with cumulative distribution function F, marginal distribution functions F_1, F_2 and finite means μ_1, μ_2. The rth L-comoment of variable X_1 with respect to variable X_2 (in this order) is defined as:

$$\lambda_{r[12]} = \mathrm{cov}(X_1, P_{r-1}^*(F_2(X_2))), r \geq 2,$$

(the version $\lambda_{r[21]}$ is defined similarly). Generally, $\lambda_{r[12]}$ and $\lambda_{r[21]}$ are not equal. Having $X_1 = X_2$, L-comoments reduce to univariate L-moments. The second to the fourth L-comoments may be regarded as alternatives to central comoments $\xi_{2[12]}, \xi_{3[12]}, \xi_{4[12]}$.

Scale-free versions of L-comoments, so-called L-comoment coefficients, are defined in similar way as L-moment coefficient of variation and L-moment ratios given by Formula (4):

$$\tau_{2[12]} = \frac{\lambda_{2[12]}}{\lambda_1^{(1)}}, \tau_{r[12]} = \frac{\lambda_{k[12]}}{\lambda_2^{(1)}}, r \geq 3.$$

Computation of population L-comoments may be simplified when variables X_1, X_2 meet certain conditions, particularly when variables are jointly distributed with affinely equivalent marginal distributions and one variable has linear regression on the other (for a detailed formulation see Proposition 3 in Serfling and Xiao (2007)).

1.2.2 Estimation of L-comoments

As it is in the case of univariate L-moments, L-comoments must be in practice estimated from an observed random sample drawn from an unknown multivariate distribution. This is made in terms of concomitants. Consider a sample $\{(x_i^1, x_i^2), 1 \leq i \leq n\}$ drawn from an unknown bivariate distribution. When the sample $\{x_1^2, ..., x_n^2\}$is sorted to a non-decreasing sequence, then the element of the sample $\{x_1^1, ..., x_n^1\}$ which is paired to the rth order statistic $x_{r:n}^2$ is called the concomitant of $x_{r:n}^2$ and denoted by $x_{[r:n]}^{12}$. The unbiased estimator of the rth L-comoment $\lambda_{r[12]}$ is defined as a linear combination of concomitants:

$$\hat{\lambda}_{r[12]} = \frac{1}{n} \sum_{r=1}^{n} w_{k:n}^{(r)} x_{[r:n]}^{12}, r \geq 2, \tag{6}$$

where:

$$w_{k:n}^{(r)} = \frac{1}{n} \sum_{j=0}^{\min\{r-1,k-1\}} (-1)^{r-j-1} \binom{r-1}{j} \binom{r-1+j}{j} \binom{n-1}{j}^{-1} \binom{k-1}{j}$$

are the weights.

1.2.3 Multivariate L-moments as L-comoment matrices

Consider a d-variate random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$. The multivariate L-moment of the first order is the vector mean:

$$\Lambda_1 = E(X_1, \ldots, X_d),$$

while the second and higher orders multivariate L-moments are defined in a matrix form with elements the rth L-comoments of variables $X_i, X_j, 1 \leq i,j \leq d$,

$$\Lambda_r = \left(\lambda_{r[ij]}\right)_{d \times d}. \tag{7}$$

The second, third and fourth multivariate L-moments $\Lambda_2, \Lambda_3, \Lambda_4$ are termed L-covariance, L-coskewness and L-cokurtosis matrices, respectively. Scale-free versions of L-comoment matrices Λ_r, $r \geq 2$, labelled as the L-comoment coefficient matrices Λ_r^* consist of L-comoment coefficients of variables $X_i, X_j, 1 \leq i,j \leq d$,

$$\Lambda_r^* = \left(\tau_{r[ij]}\right)_{d \times d}.$$

The diagonal elements of matrices Λ_r and Λ_r^* are obviously the univariate L-moments and L-moment ratios.

Multivariate L-moments of three selected bivariate distributions are presented in Table 3.

Table 3 L-moments of several selected bivariate distributions

Distribution	Joint density function and L-moments
Normal	$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$, $x, y, \mu_1, \mu_2 \in R, \sigma_1, \sigma_2 > 0, \rho \in (-1, 1)$ $\Lambda_1 = (\mu_1, \mu_2), \Lambda_2 = \frac{1}{\sqrt{\pi}} \cdot M, \Lambda_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \Lambda_4 = 0.0702 \cdot M$, where: $M = \begin{pmatrix} \sigma_1 & \rho\sigma_1 \\ \rho\sigma_2 & \sigma_2 \end{pmatrix}$
Pareto Type I	$f(x,y) = \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}\left(\frac{x}{\sigma_1} + \frac{y}{\sigma_2} - 1\right)^{-\alpha-2}, x \geq \sigma_1 > 0, y \geq \sigma_2 > 0, \alpha > 0$ $\Lambda_1 = \left(\frac{\alpha\sigma_1}{\alpha-1}, \frac{\alpha\sigma_2}{\alpha-1}\right), \Lambda_2 = \frac{1}{(\alpha-1)(2\alpha-1)} \cdot M, \Lambda_3 = \frac{\alpha+1}{(\alpha-1)(2\alpha-1)(3\alpha-1)} \cdot M, \Lambda_4 = \frac{(\alpha+1)(2\alpha+1)}{(\alpha-1)(2\alpha-1)(3\alpha-1)(4\alpha-1)} \cdot M$, where: $M = \begin{pmatrix} \alpha\sigma_1 & \sigma_1 \\ \sigma_2 & \alpha\sigma_2 \end{pmatrix}$
Pareto Type II	$f(x,y) = \frac{\alpha(\alpha+1)}{\sigma_1\sigma_2}\left(\frac{x-\mu_1}{\sigma_1} + \frac{y-\mu_2}{\sigma_2} + 1\right)^{-\alpha-2}, x > \mu_1, \mu_1 \in R, \sigma_1 > 0, y > \mu_2, \mu_2 \in R, \sigma_2 > 0, \alpha > 0$ $\Lambda_1 = \left(\mu_1 + \frac{\sigma_1}{\alpha-1}, \mu_2 + \frac{\sigma_2}{\alpha-1}\right), \Lambda_2 = \frac{1}{(\alpha-1)(2\alpha-1)} \cdot M, \Lambda_3 = \frac{\alpha+1}{(\alpha-1)(2\alpha-1)(3\alpha-1)} \cdot M, \Lambda_4 = \frac{(\alpha+1)(2\alpha+1)}{(\alpha-1)(2\alpha-1)(3\alpha-1)(4\alpha-1)} \cdot M$, where: $M = \begin{pmatrix} \alpha\sigma_1 & \sigma_1 \\ \sigma_2 & \alpha\sigma_2 \end{pmatrix}$

Source: Serfling and Xiao (2007), own construction

Multivariate L-moments are estimated by considering estimates of L-comoments, defined by Formula (6), in the matrix given in Formula (7). In the similar way, the L-comoment coefficient matrices are estimated.

1.3 Regional frequency analysis

Occurrence of extreme events, e.g., in hydrology, meteorology and climatology, among others, observed nowadays in many parts of the world may impact negatively on human society. Therefore, to reduce their impact, it is important to best estimate hiqh quantiles of a given return period. Although, in many practical applications the number of measurements is not sufficient to reliably estimate high quantiles (e.g., when annual maxima are measured), the same variable is often measured in other sites. In theses cases, RFA then provides more accurate estimates of high quantile in comparison to local approaches by taking into account data from different sites which have probability distributions similar to that site of interest. In index-flood based RFA introduced by Dalrymple (1960), a set of sites must meet a homogeneity condition, which means that all sites within a region have identical probability distributions apart from a site-specific scale factor (regions that meet this condition are termed homogeneous, otherwise they are termed heterogeneous). The multivariate quantile $\boldsymbol{Q}_i(F)$, $0 < F < 1$, at site is estimated as:

$$\widehat{\boldsymbol{Q}}_i(F) = \widehat{\boldsymbol{\mu}}_i \widehat{\boldsymbol{q}}(F),$$

where $\widehat{\boldsymbol{\mu}}_i$ corresponds to an estimate of the index-flood scale factor at site (usually estimated by sample mean or median) and $\widehat{\boldsymbol{q}}(\cdot)$ is an estimate of the regional growth curve which is a dimensionless quantile function of the probability distribution that is common to all sites in the region.

Generally, RFA consists of two main parts: 1) identification of homogeneous regions, and 2) quantile estimation. Here, the focus is on identification of homogeneous regions, i.e., groups of sites having probability distributions identical apart from a site-specific scale factor, because it is a key task in index-flood based RFA.

At first, a region must be proposed. Generally, it is recommended to form sites into groups on the basis of the site characteristics, such as the geographical location and elevation. They should not be based on at-site characteristics, because they are used for homogeneity testing as will be shown later. Several procedures have been proposed in the literature to form groups of similar sites, however, cluster analysis is the most practical method (Gordon, 1981; Everitt, 1993). When the region has been already proposed, it is desirable to decide whether it may be regarded as homogeneous, and, hence, the data from other sites may be utilized to obtain accurate estimates of high quantiles. Before executing the homogeneity test, the discordancy test should be applied to detect discordant sites.

1.3.1 L-moment discordancy test

The first step in any data analysis is to check that the data are suitable for the analysis. Two kinds of errors may occur: 1) data values are incorrect, and 2) the circumstances under which the data are ollected change over time. Sample L-moments may be used to reveal these errors. The aim of the L-moment discordancy test is to detect sites which are discordant with the group of sites as a whole.

Let's have a group of N sites, with site i having the record length n_i and sample L-comoment coefficient matrices $\Lambda_2^{*(i)}, \Lambda_3^{*(i)}, \Lambda_4^{*(i)}$. The discordancy measure is in the form:

$$D_i = \frac{1}{3}(U_i - \overline{U})^T S^{-1} (U_i - \overline{U}),$$

where:

$$U_i^T = [\Lambda_2^{*(i)}\Lambda_3^{*(i)}\Lambda_4^{*(i)}], S = \frac{1}{N-1}\sum_{i=1}^{N}(U_i - \overline{U})(U_i - \overline{U})^T, \overline{U} = \frac{1}{N}\sum_{i=1}^{N}U_i$$

(A^T denotes a transposed matrix A). A site i is regarded to be discordant if $||D_i|| > 2.6$. The sites flagged as discordant should be further checked.

1.3.2 L-moment homogeneity testing

First in the literature, the parametric multivariate L-moment homogeneity test was introduced as a generalization of the univariate Hosking and Wallis (1997) test to the multivariate case. However, this test is parametric which means that the multivariate probability distribution common to all sites must be specified. Moreover, the threshold for decision about homogeneity comes from simulations. To avoid drawbacks of the parametric test above-mentioned, Masselot, Chebana and Ouarda (2016) have introduced three alternatives, which differ in generating synthetic homogeneous regions and in the way of decision about homogeneity in comparison to the Chebana and Ouarda (2007) procedure. From all three alternatives proposed, here, the focus is only on the permutation nonparametric test which has the best performance according to the simulation study performed.

Parametric L-moment Homogeneity Test

1) Compute the statistic

$$V_{\|\cdot\|} = \left(\frac{\sum_{i=1}^{N} n_i \| \Lambda_2^{*(i)} - \overline{\Lambda}_2^{*} \|^2}{\sum_{i=1}^{N} n_i} \right)^{1/2}. \tag{8}$$

   where $\overline{\Lambda}_2^* = (\sum_{i=1}^{N} n_i \Lambda_2^{*(i)})/\sum_{i=1}^{n} n_i$ is the regional L-covariance coefficient matrix and $||\cdot||$ an arbitrary matrix norm (Chebana and Ouarda (2007) recommend the spectral matrix norm).

2) Generate a large number N_{sim} of homogeneous regions (500 regions is enough according to Chebana and Ouarda, 2007) with N sites, each having the same record length as its real-world counterpart. To get a sample with univariate margins use copulas, and to get the desired sample use the quantile function of a four-parameter kappa distribution. A copula, being very flexible in modelling the dependence structure between variables, is a multivariate distribution function whose one-dimensional margins are uniform on the interval (0, 1). Sklar's theorem (Sklar, 1959) provides the relationship between a copula C, joint distribution function H and univariate margins $F_1, ..., F_d$:

$$H(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d)) \forall x_1,\ldots,x_d \in R.$$

   Copula theory has been well developed in the literature, see e.g., Joe (1997) and Nelsen (2006) for detailed copula foundations. The regional weighted parameters of the kappa distribution are estimated using the L-moment method proposed by Hosking (1990) by fitting the kappa distribution to the regional L-moment ratios $(1, t_2^R, t_3^R, t_4^R)$, where t_k^R is a weighted mean of the at-site L-moment ratios for $k = 2,3,4$, while the regional copula parameter is obtained as a weighted mean of the at-site estimates using the at-site record lengths as weights.

3) Compute the statistic $V_{||\cdot||}^{(j)}$ defined by the Formula (8) on each of the simulated homogeneous regions, $j = 1, ..., N_{sim}$. Standardize $V_{\|\cdot\|}$ computed on the observed data in the first step by the mean μ and standard deviation σ of the computed values of $V_{||\cdot||}^{(j)}$ for a large number of simulated regions, i.e.,

$$H_{\|\cdot\|} = \frac{V_{\|\cdot\|} - \mu}{\sigma}.$$

4) Categorize the region: the region is declared to be homogeneous if $H_{\|\cdot\|} < 1$, acceptably homogeneous if $1 \leq H_{\|\cdot\|} < 2$, and definitely heterogeneous if $H_{\|\cdot\|} \geq 2$. Naturally, other measures used by Hosking and Wallis (1997) in the univariate L-moment homogeneity testing may be considered in the multivariate case to detect heterogeneity.

Nonparametric Permutation L-moment Homogeneity Test

1) Choose a significance level $\alpha \in (0,1)$.
2) Calculate $V_{\|\cdot\|}$ defined by the Formula (8) on the observed data as in the first step of the parametric test.
3) Generate a large number N_{sim} of homogeneous regions, which means to reassign randomly the pooled data between N sites while preserving the real-world at-site record lengths.
4) Compute the statistic $V^{(j)}_{\|\cdot\|}$ defined by the Formula (8) on each of the simulated regions, $j = 1, ..., N_{sim}$.
5) Compute the *p-value* given by:

$$p - value = \frac{1}{N_{sim}} \#\{V^{(j)}_{\|\cdot\|} > V_{\|\cdot\|}\}\,. \tag{9}$$

The null hypothesis of homogeneity is rejected if $p - value < \alpha$.

Although RFA has been traditionally used for analysis of extreme natural phenomena, it may be also employed in other fields in which extremes appear. In particular, it seems that modelling and estimation in finance, in which the interest in multivariate heavy-tailed distributions has increased, could be improved by using RFA.

2 RESULTS

In this section, results of specific applications of L-moments in two main fields of statistical inference are presented.

2.1 Estimation of GPD parameters

The choice of an appropriate estimation method of the GPD parameters is solved in this section. The three-parametric GPD with parameters ξ (location), σ (scale) and k (shape) has cumulative distribution function in the form:

$$F(x) = \begin{cases} 1 - [1 - \frac{k(x-\xi)}{\sigma}]^{1/k} & ,k \neq 0 \\ 1 - e^{-\frac{x-\xi}{\sigma}}, k = 0. \end{cases}.$$

The L-moments estimates are compared to estimates obtained by the moment and maximum likelihood methods. Note that population L-moments of all orders exist for $k > -1$. Matching the first three population L-moments to their sample counterparts and solving the system of equations given in (5), L-moments based parameter estimates are obtained:

$$\hat{k} = \frac{1-3t_3}{1+t_3}, \hat{\sigma} = l_2(\hat{k}+1)(\hat{k}+2), \hat{\xi} = l_1 - \frac{\hat{\sigma}}{\hat{k}+1}.$$

Analogously, moments based estimates are given by:

$$g = \frac{2(1-\hat{k})\sqrt{1+2\hat{k}}}{1+3\hat{k}}, \hat{\sigma} = s(\hat{k}+1)\sqrt{1+2\hat{k}}, \hat{\xi} = \bar{x} - \frac{\hat{\sigma}}{\hat{k}+\hat{\sigma}},$$

where $\bar{x}, s^2, g$ are sample mean, variance and skewness, respectively. First, the shape parameter k must be estimated by numerically solving the first equation. In the case of the maximum likelihood method, the location parameter ξ cannot be obtained, because the likelihood function is not bounded with respect to ξ, hence, the minimum value of the sample data is used as its estimate (Singh and Guo, 1995). The estimates of σ and k are achieved by solving equations:

$$\sum_{i=1}^{n} \frac{(x_i - \xi)/\sigma}{1 - k(x_i - \xi)/\sigma} = \frac{n}{1-k}, \sum_{i=1}^{n} \ln\left[1 - k(x_i - \xi)/\sigma\right] = -nk.$$

According to Šimková and Picek (2016) this study is focused only on values of the shape parameter k in the range $-0.4 \le k \le 0.4$ being typical for environmental applications. For each combination of the sample size n, $n \in \{20, 50, 100\}$, and the shape parameter k, $k \in \{-0.4, -0.2, 0, 0.2, 0.4\}$ 1 000 times a sample from the GPD is drawn, while the parameters of location and scale are fixed $\xi = 0$, $\sigma = 1$. The estimation methods are compared each other according to the sample mean squared error (MSE):

$$MSE = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2.$$

Table 4 Parameter estimates by moment (MM), L-moment (LM) and maximum likelihood (ML) methods: sample mean over 1 000 simulations (first row) and sample MSE (second row)

	$n = 20$			$n = 50$			$n = 100$		
	MM	LM	ML	MM	LM	ML	MM	LM	ML
ξ	0.661	−0.029	0.053	0.621	−0.014	0.020	0.578	−0.010	0.010
	0.894	0.015	*0.006*	0.607	0.005	*0.001*	0.444	0.003	*0.000*
σ	2.288	1.160	1.224	2.064	1.075	1.071	1.927	1.046	1.032
	3.456	*0.270*	0.347	2.091	0.089	*0.074*	1.320	0.047	*0.034*
$k = -0.4$	0.051	−0.255	−0.240	−0.082	−0.334	−0.351	−0.139	−0.360	−0.379
	0.231	*0.100*	0.182	0.112	*0.039*	0.048	0.075	*0.021*	*0.021*
ξ	0.299	−0.015	0.053	0.235	−0.006	0.020	0.189	−0.004	0.010
	0.200	0.011	*0.006*	0.108	0.004	*0.001*	0.068	0.002	*0.000*
σ	1.642	1.103	1.213	1.428	1.041	1.0671	1.322	1.021	1.030
	0.718	*0.206*	0.300	0.290	0.068	*0.065*	0.157	0.035	*0.029*
$k = -0.2$	0.121	−0.108	−0.028	−0.008	−0.166	−0.147	−0.062	−0.183	−0.176
	0.137	*0.086*	0.163	0.051	*0.031*	0.039	0.028	*0.016*	0.017
ξ	0.121	−0.013	0.051	0.061	−0.004	0.020	0.044	−0.003	0.010
	0.065	0.010	*0.005*	0.032	0.003	*0.001*	0.019	0.001	*0.000*
σ	1.357	1.085	1.225	1.175	1.025	1.061	1.114	1.016	1.035
	0.307	*0.191*	0.307	0.095	0.060	*0.058*	0.045	0.029	*0.026*
$k = 0$	0.221	0.059	0.190	0.100	0.014	0.055	0.063	0.008	0.028
	0.096	*0.087*	0.168	0.029	*0.027*	0.032	0.014	*0.013*	0.014
ξ	0.037	−0.006	0.051	0.014	−0.002	0.019	0.005	−0.001	0.010
	0.033	0.008	*0.005*	0.015	0.003	*0.001*	0.009	0.001	*0.000*
σ	1.187	1.056	1.220	1.076	1.021	1.067	1.039	1.009	1.032
	0.165	*0.152*	0.251	0.054	0.051	*0.050*	0.027	0.026	*0.023*
$k = 0.2$	0.345	0.240	0.416	0.254	0.212	0.269	0.226	0.204	0.234
	0.070	0.078	0.157	*0.021*	0.025	0.028	*0.010*	0.013	0.011
ξ	0.010	−0.004	0.051	0.003	−0.002	0.019	0.000	−0.001	0.010
	0.020	0.007	*0.005*	0.008	0.002	*0.001*	0.005	0.001	*0.000*
σ	1.116	1.051	1.209	1.043	1.020	1.072	1.021	1.009	1.036
	0.133	0.142	0.197	*0.046*	0.049	*0.046*	0.023	0.025	*0.020*
$k = 0.4$	0.500	0.431	0.634	0.434	0.410	0.482	0.415	0.403	0.442
	0.070	0.088	0.145	*0.022*	0.029	0.028	*0.011*	0.014	*0.011*

Source: Own construction

The smallest the MSE the best the estimator is. Computations are executed in the software R (R Core Team, 2014).

Table 4 compares performance of the moments, L-moments and maximum likelihood methods (the minimum MSE value is highlighted by italics). It can be concluded that the maximum likelihood method provides the best estimate of the location parameter, and also of the scale parameter for moderate sample sizes $n = 50, 100$, while the L-moment method outperforms the moment method for small sample size $n = 20$. When estimating the shape parameter, the L-moment based estimator is recommended for heavier tails ($k \leq 0$), while the moment method yields estimates with the smallest MSE for light tails ($k > 0$).

2.2 Nonparametric homogeneity testing in bivariate RFA of extreme precipitation events

Bivariate parametric homogeneity testing based on L-moments has been already applied to data observed at meteorological stations located in the area of the Czech Republic in the study of Šimková, Picek and Kyselý (in preparation). They found out that six regions formed in the area may be regarded as homogeneous with accordance to bivariate distribution function with components the 1- and 5-day maximum annual precipitation totals. Hence, this finding justifies to use data from an entire region for estimating quantiles in any target site in region. However, the procedure used required the user interventions: a bivariate copula specification, and kappa distribution and copula parameters estimation. Here, the homogeneity of regions is also checked by the nonparametric permutation test proposed recently by Masselot, Chebana and Ouarda (2016), which is easy to apply. We want to test the null hypothesis:

H_0 Region is homogeneous,

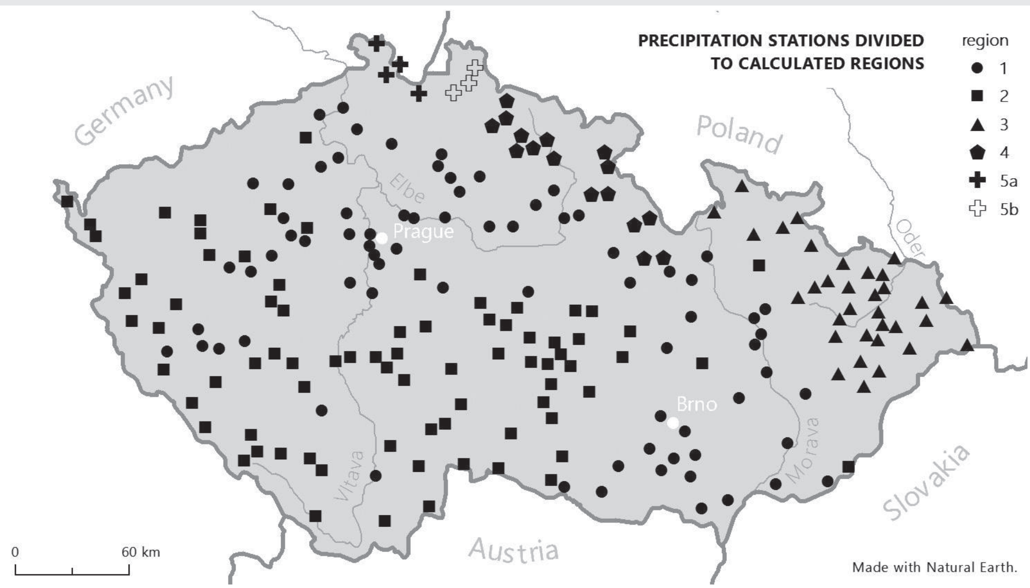
against the alternative:

H_1 Region is not homogeneous,

on the 5% significance level.

Maximum annual 1- and 5-day precipitation amounts measured mostly from 1961 to 2007 at 210 stations covering the area of the Czech Republic are used as the input dataset. The data were provided

Figure 1 Delineation of stations into six regions



Source: Šimková (accepted)

by the Czech Hydrometeorological Institute (CHMI), where they underwent basic quality checking. Kyselý (2009) also checked thoroughly the data for errors and missing readings. Delineation of the stations to regions shown in Figure 1 is that presented first in the study of Šimková (accepted). Basic information concerning the datasets for each region is summarized in Table 5. See Šimková (accepted) for more description of regions studied.

Table 5 Information on the input datasets

Region	1	2	3	4	5a	5b
Number of stations	75	79	33	16	4	4
Overall record length	3 438	3 633	1 508	719	188	141
Minimal record length (years)	33	37	33	36	47	47
Maximal record length (years)	47	47	47	47	47	47
Average record length (years)	45.8	46.0	45.7	44.9	47	47
Altitude range (m a.s.l.)	[150, 400]	[410, 1 118]	[220, 1 490]	[255, 572]	[315, 440]	[398, 778]
Average altitude (m a.s.l.)	270.1	550.3	411.1	412.6	361	523.3

Source: Šimková, Picek and Kyselý (in preparation)

The problem of determining discordant sites and their retention in regions have been already discussed by Šimková, Picek and Kyselý (in preparation). To estimate p-values given by Formula (9), 500 synthetic regions were generated by permuting bivariate data between sites, while the values of $V_{||\cdot||}$ have been already calculated on real observed data by Šimková, Picek and Kyselý (in preparation). Table 6 shows the values of $V_{||\cdot||}$, and compares the results of parametric and nonparametric homogeneity testing via the heterogeneity measures and p-values obtained. Values of the heterogeneity measure $H_{||\cdot||}$ are those presented by Šimková, Picek and Kyselý (in preparation). The parametric test gives evidence about homogeneity of all regions because $H_{||\cdot||}$ values are less than 2, while the nonparametric version rejects the null hypothesis H_0 of homogeneity for region 1 on the 5% significance level.

Table 6 Homogeneity testing results

Region	$V_{||\cdot||}$	$H_{||\cdot||}$	*p-value*
1	0.0603	1.4884	0.006
2	0.0570	1.1316	0.226
3	0.0536	–1.3857	0.994
4	0.0551	0.7111	0.090
5a	0.0181	–1.5305	0.976
5b	0.0278	–0.7635	0.770

Source: Šimková, Picek and Kyselý (in preparation), own construction

Hence, region 1 should be redefined. Because Šimková, Picek and Kyselý (in preparation) proposed region 1 as a unification of three smaller regions labelled 1a, 1b and 1c, presented in the study of Kyselý, Gaál and Picek (2011), the original delineation could be now considered. Homogeneity of these smaller regions has been checked again and they have finally met the homogeneity condition.

Table 7 Homogeneity testing results for redefined region 1

Region	$V_{\|\cdot\|}$	$H_{\|\cdot\|}$	*p-value*
1a	0.0364	–2.4479	0.996
1b	0.0590	0.8400	0.116
1c	0.0426	–0.8644	0.792

Source: Own construction

CONCLUSION

Alternatives to traditionally used moments and comoments labelled L-moments, which exist under only finite mean assumptions, have been introduced in the paper presented. L-moments, being measures of shape of a probability distribution, may be used to describe a probability distribution and to summarize sample data. Population L-moments of several selected both univariate and bivariate distributions have been also presented. The paper also shows selected already established L-moments based techniques and implements them in particular tasks of statistical inference.

The problem of estimating GPD parameters has been resolved. In a small comparison simulation study, in which three parameters of the GPD were estimated, it has been shown that the method based on L-moments outperforms other usually used estimation methods, such as the maximum likelihood and moments methods. This happens particularly for heavier tailed distributions and small to moderate samples. These results are consistent with those obtained for other probability distributions.

The benefits of the nonparametric test based on L-moments have been applied for regions formed in the area of the Czech Republic. The results obtained by the nonparametric test have confirmed those obtained by the parametric one (except one region), but in a much shorter time and without estimating parameters and selection of a suitable bivariate copula family, which is substantially more advantageous. Although RFA has been traditionally used for analysis of natural phenomena, such as floods and precipitation, nothing prevents to use it also for example in finance or economics, because extremes also appear there.

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References

BEN AISSIA, M. A., CHEBANA, F., OUARDA, T. B. M. J., BRUNEAU, P., BARBET, M. Bivariate Index-flood Model for a Northern Case Study. *Hydrological Sciences Journal*, 2014. DOI: 10.1080/02626667.2013.87517.

BÍLKOVÁ, D. Alternative Means of Statistical Data Analysis: L-Moments and TL-Moments of Probability Distributions [online]. *Statistika: Statistics and Economy Journal*, 2014, 2, pp. 77–94.

CHEBANA, F. AND OUARDA, T. B. M. J. Multivariate L-moment homogeneity test. *Water Resources Research*, 2007, 43. DOI: 10.1029/2006WR005639.

CHEBANA, F. AND OUARDA, T. B. M. J. Index-flood Based Multivariate Regional Frequency Analysis. *Water Resources Research*, 2009, 45. DOI: 10.1029/2008WR007490.

CHEBANA, F., OUARDA, T. B. M. J., FAGHERAZZI, L., BRUNEAU, P., BARBET, EL ADLOUNI, S., LATRAVERSE, M. Multivariate Homogeneity Testing in a Northern Case Study in the Province of Quebec, Canada. *Hydrological Processes*, 2009, 23(12), pp. 1690–1700.

CHEN, Y. D., HUANG, G., SHAO, Q., XU, C.-Y. Regional Analysis of Low Flow using L-moments for Dongjiang Basin, South China. *Hydrological Sciences Journal*, 2006, 51(6), pp. 1051–1064.

DALRYMPLE, T. Flood Frequency Analyses. *Water Supply Paper 1543-A*, U.S. Geological Survey.

DELICADO, P. AND GORIA, M. N. A Small Sample Comparison of Maximum Likelihood, Moments and L-moments Methods for the Asymmetric Exponential Power Distribution. *Computational Statistics and Data Analysis*, 2008, 52(3), pp. 1661–1673.

ELAMIR, E. A. H. AND SEHEULT, A. H. Trimmed L-moments. *Computational Statistics and Data Analysis*, 2003, 43(3), pp. 299–314.

EVERITT, B. S. *Cluster Analysis.* London: Edward Arnold, 1993.

GORDON, A. D. *Classification: Methods for the Exploratory Analysis of Multivariate Data*. London: Chapmann & Hall, 1981.

HOSKING, J. R. M. L-moments: Analysis and Estimation of Distributions Using Linear Combinations of Order Statistics. *Journal of Royal Statistical Society (Series B)*, 1990, 52, pp. 105–124.

HOSKING, J. R. M. AND WALLIS, J. R. Parameter and Quantile Estimation for the Generalized Pareto Distribution. *Technometrics*, 1987, 29, pp. 339–349.

HOSKING, J. R. M. AND WALLIS, J. R. *Regional Frequency Analysis: An Approach Based on L-moments*. Cambridge: Cambridge University Press, 1997.

HOSKING, J. R. M., WALLIS, J. R., WOOD, E. F. Estimation of the Generalized Extreme-value Distribution by the Method of Probability-weighted Moments. *Technometrics*, 1985, 27, pp. 251–261.

JOE, H. *Multivariate Models and Dependence Concepts*. London: Chapman & Hall, 1997.

KYSELÝ, J. Trends in Heavy Precipitation in the Czech Republic over 1961–2005. *International Journal of Climatology*, 2009, 29, pp. 1745–1758.

KYSELÝ, J. AND PICEK, J. Regional Growth Curves and Improved Design Value Estimates of Extreme Precipitation Events in Czech Republic. *Climate Research*, 2007, 33, pp. 243–255.

KYSELÝ, J., GAÁL, L., PICEK, J. Comparison of Regional and At-site Approaches to Modelling Probabilities of Heavy Precipitation. *International Journal of Climatology*, 2011, 31, pp. 1457–1472.

KYSELÝ, J., PICEK, J., HUTH, R. Formation of Homogeneous Regions for Regional Frequency Analysis of Extreme Precipitation Events in the Czech Republic. *Studia Geophysica et Geodaetica*, 2006, 51, pp. 327–344.

MARTINS, E. S. AND STEDINGER, J. R. Generalized Maximum-likelihood Generalized Extreme-value Quantile Estimators for Hydrologic Data. *Water Resources Research*, 2000, 36(3), 737–744.

MASSELOT, P., CHEBANA, F., OUARDA, T. B. M. J. Fast and Direct Nonparametric Procedures in the L-moment Homogeneity Test. *Stochastic Environmental Research and Risk Assessment*, 2016. DOI: 10.1007/s00477-016-1248-0.

MUDHOLKAR, G. S., HUTSON, A. D. LQ-moments: Analogs of L-moments. *Journal of Statistical Planning and Inference*, 1998, 71, pp. 191–208.

NELSEN, R. B. *An Introduction to Copulas*. New York: Springer-Verlag New York, 2006.

NOTO, L. V., LA LOGGIA, G. *Use of L-Moments Approach for Regional Flood Frequency Analysis in Sicily, Italy*. Water Resources Management, 2009, 23, pp. 2207–2229.

R CORE TEAM. R: A Language and Environment for Statistical Computing (Version 3.3.1) [Computer software]. Vienna, Austria: R Foundation for Statistical Computing, 2016.

REQUENA, A. I., MEDIERO, L., GARROTEL, L. A Complete Procedure for Multivariate Index-flood Model Application. *Journal of Hydrology*, 2016, 535, pp. 559–580.

SERFLING, R. AND XIAO, P. A Contribution to Multivariate L-comoments: L-comoment Matrices. *Journal of Multivariate Analysis*, 2007, 98, pp. 1765–1781.

SINGH, V. P. AND GUO, H. Parameter Estimation for 3-parameter Generalized Pareto Distribution by the Principle of Maximum Entropy (POME). *HydrologicaL Sciences Journal*, 1995, 40(2), pp. 165–181.

SKLAR, A. Fonctions de Rèpartition à *n* Dimensions et Leurs Marges. *Publications de l'Institut de Statistique de L'Universitè de Paris*, 1959, 8, pp. 229–231.

ŠIMKOVÁ, T. L-moment Homogeneity Test in Trivariate Regional Frequency Analysis of Extreme Precipitation Events. *Meteorological Applications* (accepted).

ŠIMKOVÁ, T. AND PICEK, J. A Comparison of L-, Lq-, Tl-Moment and Maximum Likelihood High Quantile Estimates of the Gpd and Gev Distribution. *Communications in Statistics – Simulation and Computation*, 2016. DOI: 10.1080/03610918.2016.1188206.

ŠIMKOVÁ, T., PICEK, J., KYSELÝ, J. *Bivariate Regional Frequency Analysis of Extreme Precipitation Events in the Czech Republic* (in preparation).

VIGLIONE, A., LAIO, F., CLAPS, P. A *Comparison of Homogeneity Tests for Regional Frequency Analysis.* Water Resources Research, 2007, 43(3). DOI:10.1029/2006WR005095.