Recursive Estimation of Volatility for High Frequency Financial Data

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Abstract

The paper deals with recursive estimation of financial time series with conditional volatility. It surveys the recursive methodology suggested in Hendrych and Cipra (2018) and adjusts it for various alternatives of GARCH models which are usual in financial practice. Such a recursive approach seems to be suitable for the dynamic estimation with high-frequency data. The paper verifies the applicability of recursive algorithms of particular models to high-frequency data from the Czech environment, particularly in the context of risk prediction.

Keywords	***JEL code***
GARCH, high-frequency financial time series, recursive estimation, risk prediction, volatility	*C32, C51, C58*

INTRODUCTION

In the case of financial time series modeling, models with conditional heteroscedasticity GARCH are currently preferred in practice. They present the most powerful tool for routine modeling of financial time series. In practice, these models are commonly estimated using static (off-line or batch) methods, e.g., the maximum likelihood estimation. However, the application of the static methods to high-frequency data, such as stock market data, is problematic or even impossible. As an example, in the case of stock prices, where minute or even more frequent data are encountered, the use of static methods would be computationally impossible. For this reason, recursive methods are preferred for high-frequency data.

In literature, there have already been proposed recursive algorithms for GARCH model estimation, e.g. Kierkegaard et al. (2000), Aknouche and Guerbyenne (2006) or Hendrych and Cipra (2018, 2019). The recursive methodology suggested in Hendrych and Cipra (2018) can be adjusted for various types of GARCH models (see the recursive algorithms for models GJR-GARCH, IGARCH and EGARCH in Section 2). The aim of this paper is to verify the applicability of these recursive algorithms to real high-frequency data from the Czech environment. In particular, the risk prediction potential of this recursive methodology is investigated using specific methods (MAPE criterion, realized volatility).

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The paper is organized as follows. Section 1 presents models with conditional heteroskedasticity that can be used for modeling in finance, namely the modifications of the GARCH model. At the same time, this section presents the algorithm for recursive estimation of volatility for the GARCH model. Section 2 focuses on the description of recursive formulas for modifications of the GARCH model, such as GJR-GARCH, IGARCH and EGARCH, and briefly comments results of a simulation study. In Section 3, the application of the proposed algorithms to real high-frequency data is presented including the risk prediction analysis. Finally, the last section summarizes conclusions.

1 GARCH MODELS AND RECURSIVE ESTIMATION OF VOLATILITY

When working with time series, there exist several ways how to model them. However, when dealing with financial time series, the usual data generating mechanism depends on the first and second conditional moments, see Cipra (2020). Thus, these time series are assumed in the following form:

$$y_t = \mu_t + e_t = \mu_t + \sigma_t \varepsilon_t, \tag{1}$$

where μ_t represents the conditional mean, σ_t is the square root of the conditional variance and ε_t's are independent, identically distributed random variables with zero mean and unit variance. Our primary aim is to model the conditional variance and, in particular, to find recursive algorithms for its estimation in time.

1.1 GARCH models in financial practice

In literature, many different approaches to modeling the conditional variance have been considered so far. However, the strongest tool for financial time series modeling, which has not yet been overcome, are GARCH models.

1.1.1 GARCH model

The most important model from this class of models is the GARCH model proposed by Bollerslev (1986), where the equation for the conditional variance has the following form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{2}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i > 0$, $\beta_j \geq 0$ for $j > 0$. These are the conditions to ensure positivity of the conditional variance. The stationarity is provided by fulfillment of an additional condition $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$. By adding the lagged values of the conditional variance into the equation, the model can more successfully capture volatility clustering, which is typical for financial time series.

Although the GARCH model is undoubtedly the most popular of the models with conditional heteroscedasticity, we encounter many modifications of this basic model in financial practice. These modifications aim to eliminate some of the drawbacks of the GARCH model and improve its properties so that it is as close as possible to the real behavior of the data (see below).

1.1.2 IGARCH model

One of the simplest extension is the integrated GARCH model with orders p, q, usually denoted as the IGARCH(p, q) model (see Engle and Bollerslev, 1986). The only difference consists in a stricter parameter constraint:

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1. \tag{3}$$

Formula (3) causes the non-existence of the unconditional variance. The impact of current information persists in conditional volatility forecasts for long horizons. For instance, the popular EWMA model is a special case of the IGARCH(1,1) model.

1.1.3 GJR-GARCH model

In order to capture the leverage effect, another modification was proposed by Glosten, Jagannathan and Runkle (1993). The volatility equation of GJR-GARCH(p, q) model has the following form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} (\alpha_i + \gamma_i I_{t-i}^-) e_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{4}$$

where I_{t-i}^- denotes an indicator, which is equal to 1 if $e_{t-i} < 0$ and equal to 0 otherwise.

The sufficient conditions for σ_t^2 being positive are $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\alpha_i + \gamma_i \geq 0$ for $i > 0$ and $\beta_j \geq 0$ for $j > 0$. There is no general set of conditions to ensure that the time series is stationary. The new parameter γ_i, which regulates the different effect of e_{t-i} according to its sign. If e_{t-i} is negative, the impact is higher and the leverage effect is present.

1.1.4 EGARCH model

Another model including the leverage effect is the exponential GARCH model (EGARCH(p, q)) proposed by Nelson (1991). We apply it in the form:

$$\ln(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{p} \alpha_i \ln(\sigma_{t-i}^2) + \sum_{j=0}^{q} \delta_j \frac{y_{t-1-j}}{\sqrt{\sigma_{t-1-j}^2}} + \sum_{k=0}^{q} \gamma_k \left(\frac{|y_{t-1-k}|}{\sqrt{\sigma_{t-1-k}^2}} - \sqrt{\frac{2}{\pi}} \right), \tag{5}$$

where α_0, α_i, δ_j and γ_k are parameters. Due to the logarithmic transformations in (5), the positivity of volatility is fulfilled without any conditions imposed on the parameters.

1.2 State-space representation of GARCH models

For some of these models, it is possible to use algorithms implemented in various software systems such as EViews or R. However, this approach cannot be applied, for example, to high-frequency data such as stock market prices or index levels since the volume of such data may be enormous and real-time parameter estimation is not possible. For this reason, a recursive approach is more appropriate.

Several articles already dealt with a derivation of recursive algorithms for GARCH models, e.g., Kierkegaard et al. (2000), Aknouche and Guerbyenne (2006), Gerencsér et al. (2010), and Hendrych and Cipra (2018). These articles primarily focused on the GARCH model. In this paper, we will follow the procedure proposed by Hendrych and Cipra (2018), which is based on the general recursive algorithms, see also Ljung and Söderström (1983) or Ljung (1999). The procedures of this type are called the recursive pseudo-linear regression or the prediction error method.

In order to obtain a recursive algorithm, which could be used to estimate parameters in the basic GARCH model (2), it is necessary to transform the volatility equation into a vector form. Furthermore, the conditional mean will be considered equal to zero for simplicity. The modified form of the conditional volatility Formula (2) is:

$$\boldsymbol{\varphi}_t^T(\boldsymbol{\theta})\, \boldsymbol{\theta} = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \boldsymbol{\varphi}_{t-j}^T(\boldsymbol{\theta})\, \boldsymbol{\theta}, \tag{6}$$

where $\boldsymbol{\theta} = (\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)^T$ is the vector of model parameters, and $\boldsymbol{\varphi}_t(\boldsymbol{\theta}) = (1, y_{t-1}^2, \ldots, y_{t-p}^2, \boldsymbol{\varphi}_{t-1}^T(\boldsymbol{\theta})\, \boldsymbol{\theta}, \ldots, \boldsymbol{\varphi}_{t-q}^T(\boldsymbol{\theta})\, \boldsymbol{\theta})^T$ is the vector constructed so that the volatility equation holds. Since the conditional mean is zero, the terms e_{t-i}^2 were replaced by y_{t-i}^2.

The most important target is to construct the estimates $\hat{\boldsymbol{\theta}}$ recursively in time. Hendrych and Cipra (2018) suggest a self-weighted approach based on the minimization of a loss function corresponding to the weighted log-likelihood approach. The final algorithm has the following form:

$$\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t-1} + \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t(y_t^2 - \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}, \tag{7}$$

$$\hat{\mathbf{P}}_t = \frac{1}{\lambda_t}\left[\hat{\mathbf{P}}_{t-1} - \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}\right], \tag{8}$$

$$\hat{\boldsymbol{\varphi}}_{t+1} = (1, y_t^2, \dots, y_{t+1-p}^2, \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t, \dots, \hat{\boldsymbol{\varphi}}_{t+1-q}\hat{\boldsymbol{\theta}}_{t+1-q})^T, \tag{9}$$

$$\hat{\boldsymbol{\psi}}_{t+1} = \hat{\boldsymbol{\varphi}}_{t+1} + \sum_{j=1}^{q}\hat{\beta}_{j,t}\hat{\boldsymbol{\psi}}_{t+1-j} \tag{10}$$

for $t \in \mathbb{N}$, where $\hat{\mathbf{P}}_t$ is a $(1 + p + q) \times (1 + p + q)$ square matrix.

Several issues need to be addressed. The first of them is the choice of weights $\{\lambda_t\}$. One of the possible options is the application of a recursive formula:

$$\lambda_t = \tilde{\lambda}\lambda_{t-1} + (1 - \tilde{\lambda}),\ t \in \mathbb{N}, \tag{11}$$

as suggested in Ljung and Söderström (1983). A recommended choice of constants $\tilde{\lambda}$ and λ_0 is $\tilde{\lambda} = 0.99$ and $\lambda_0 = 0.95$. Another important choice is setting the initial estimates of the vector of parameters and some other quantities. Different options may be appropriate for each situation. One of the possibilities is to set $\hat{\boldsymbol{\theta}}_0 = (\frac{1}{n}\sum_{i=1}^{n} y_{1-i}^2[1 - (p + q)\eta], \eta, \dots, \eta)^T$, where η is a small positive constant satisfying $(p + q)\eta < 1$ for a suitable n, $\hat{\mathbf{P}}_0 = c\mathbf{I}$, where c is a suitable positive constant (e.g., c = 10^2 for this model), which ensures that the initial estimates are less influential and there is a faster convergence to the actual vector of parameters, $\hat{\boldsymbol{\varphi}}_1 = (1, y_{1-p}^2, \dots, y_0^2, k, \dots, k)^T$ with k equal to a small positive constant, and finally $\hat{\boldsymbol{\psi}}_1 = \hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\psi}}_i = \mathbf{0}$ for $i = -q + 2, \dots, 0$. If the values $y_{1-p}^2, \dots, y_0^2$ are not known, one can assume them to be equal to zero.

In addition to these choices, one can extend the proposed algorithm with a mechanism how to ensure the positivity of the conditional variance and the stationarity. This is achieved by taking the estimate at time t, according to its obtained values, as $\hat{\boldsymbol{\theta}}_t$ if $\hat{\boldsymbol{\theta}}_t \in \mathbf{D}_S$ and as $\hat{\boldsymbol{\theta}}_{t-1}$ if $\hat{\boldsymbol{\theta}}_t \notin \mathbf{D}_S$, where $\mathbf{D}_S$ is the set of vectors $\boldsymbol{\theta}$ satisfying the conditions imposed on the parameters to ensure positivity and stationarity. In the case of the GARCH model, $\mathbf{D}_S = \{\boldsymbol{\theta} \in \mathbb{R}^{p+q+1} \mid \alpha_0 > 0, \alpha_1, \dots, \alpha_p \geq 0, \beta_1, \dots, \beta_q \geq 0, \sum_{i=1}^{p}\alpha_1 + \sum_{j=1}^{q}\beta_j < 1\}$. If the estimate lies outside this set, it is ignored and the previous estimate is considered instead.

2 RECURSIVE ESTIMATION OF SELECTED GARCH MODELS

In this section, recursive algorithms for estimating the parameters of various modifications of the GARCH model from Section 1 will be presented. For the GJR-GARCH model, just a simple modification of the basic algorithm is needed. In other cases, major changes are necessary.

2.1 Recursive estimation of GJR-GARCH model

In this case the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}_t(\boldsymbol{\theta})$ are modified into:

$$\boldsymbol{\theta} = (\alpha_0, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q, \gamma_1, \dots, \gamma_p)^T, \tag{12}$$

$$\boldsymbol{\varphi}_t(\boldsymbol{\theta}) = (1, y_{t-1}^2, \dots, y_{t-p}^2, \boldsymbol{\varphi}_{t-1}^T(\boldsymbol{\theta})\,\boldsymbol{\theta}, \dots, \boldsymbol{\varphi}_{t-q}^T(\boldsymbol{\theta})\,\boldsymbol{\theta}, y_{t-1}^2 I_{t-1}^-, \dots, y_{t-p}^2 I_{t-p}^-)^T, \tag{13}$$

Similarly as in GARCH:

$$\psi_t(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} [\boldsymbol{\varphi}_t^T(\boldsymbol{\theta})\, \boldsymbol{\theta}] = \boldsymbol{\varphi}_t(\boldsymbol{\theta}) + \sum_{j=1}^{q} \beta_j \psi_{t-1}(\boldsymbol{\theta}). \tag{14}$$

Hence the corresponding estimation algorithm coincides with the one for the GARCH model except for the equation for $\hat{\boldsymbol{\varphi}}_{t+1}$. Namely,

$$\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t-1} + \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t(y_t^2 - \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}, \tag{15}$$

$$\hat{\mathbf{P}}_t = \frac{1}{\lambda_t}\left[\hat{\mathbf{P}}_{t-1} - \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}\right], \tag{16}$$

$$\hat{\boldsymbol{\varphi}}_{t+1} = (1, y_t^2, \dots, y_{t+1-p}^2, \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t, \dots, \hat{\boldsymbol{\varphi}}_{t+1-q}^T\hat{\boldsymbol{\theta}}_{t+1-q}, y_t^2 I_t^-, \dots, y_{t+1-p}^2 I_{t+1-p}^-)^T, \tag{17}$$

$$\hat{\boldsymbol{\psi}}_{t+1} = \hat{\boldsymbol{\varphi}}_{t+1} + \sum_{j=1}^{q} \hat{\beta}_{j,t}\hat{\boldsymbol{\psi}}_{t+1-j} \tag{18}$$

for $t \in \mathbb{N}$.

Also the initial estimates may be constructed in the similar way as in the case of the GARCH model. One can take $\hat{\boldsymbol{\theta}}_0 = (\frac{1}{n}\sum_{i=1}^{n} y_{1-i}^2[1 - (p+q)\eta], \eta, \dots, \eta, 0, \dots, 0)^T$, where η is a small positive constant satisfying $(p+q)\eta < 1$ for a suitable n, $\hat{\mathbf{P}}_0 = c\mathbf{I}$, where c is a suitable positive constant, $\hat{\boldsymbol{\varphi}}_1 = (1, y_{1-p}^2, \dots, y_0^2, k, \dots, k, 0, \dots, 0)^T$, with k equal to a small positive constant, $\hat{\boldsymbol{\psi}}_1 = \hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\psi}}_i = \mathbf{0}$ for $i = -q+2, \dots, 0$.

2.2 Recursive estimation of IGARCH model

As stated above, this model differs from the GARCH model by the condition $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1$. In order to include this condition directly into the volatility equation, one can rewrite it to:

$$\sigma_t^2 = y_{t-p}^2 + \alpha_0 + \alpha_1(y_{t-1}^2 - y_{t-p}^2) + \dots + \alpha_{p-1}(y_{t-p+1}^2 - y_{t-p}^2) + \beta_1(\sigma_{t-1}^2 - y_{t-p}^2) + \dots + \beta_q(\sigma_{t-q}^2 - y_{t-p}^2), \tag{19}$$

i.e., in the vector form:

$$\sigma_t^2(\boldsymbol{\theta}) = y_{t-p}^2 + \boldsymbol{\varphi}_t^T(\boldsymbol{\theta})\boldsymbol{\theta}, \tag{20}$$

where the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}_t(\boldsymbol{\theta})$ are such that (20) holds. Since the expression for $\sigma_t^2(\boldsymbol{\theta})$ was changed, it is necessary to derive the recursive algorithm newly. The derivation runs in a similar way as for the GARCH model. The final recursive algorithm can be written as:

$$\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t-1} + \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t(y_t^2 - y_{t-p}^2 - \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(y_{t-p}^2 + \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}, \tag{21}$$

$$\hat{\mathbf{P}}_t = \frac{1}{\lambda_t}\left[\hat{\mathbf{P}}_{t-1} - \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t(y_{t-p}^2 + \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})^2}\right], \tag{22}$$

$$\hat{\boldsymbol{\varphi}}_{t+1} = (1, y_t^2 - y_{t-p+1}^2, \dots, y_{t+2-p}^2 - y_{t-p+1}^2, \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t + y_{t-p}^2 - y_{t-p+1}^2, \dots, \hat{\boldsymbol{\varphi}}_{t+1-q}^T\hat{\boldsymbol{\theta}}_{t+1-q} + y_{t+1-q-p}^2 - y_{t-p+1}^2)^T, \tag{23}$$

$$\hat{\boldsymbol{\psi}}_{t+1} = \hat{\boldsymbol{\varphi}}_{t+1} + \sum_{j=1}^{q} \hat{\beta}_{j,t} \hat{\boldsymbol{\psi}}_{t+1-j} \tag{24}$$

for $t \in \mathbb{N}$.

Again the initial can be set analogously to the case of the GARCH model. That means $\hat{\boldsymbol{\theta}}_0 = (\frac{1}{n}\sum_{i=1}^{n} y_{1-i}^2[1-(p-1+q)\eta], \eta, \dots, \eta, \eta, \dots, \eta)^T$, where η is a small positive constant satisfying $(p-1+q)\eta < 1$ for a suitable n, $\hat{\mathbf{P}}_0 = c\mathbf{I}$, where c is a suitable positive constant, $\hat{\boldsymbol{\varphi}}_1 = (1, y_{2-p}^2 - y_{1-p}^2, \dots, y_0^2 - y_{1-p}^2, k, \dots, k)^T$, where k equals to a small positive constant, $\hat{\boldsymbol{\psi}}_1 = \hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\psi}}_i = \mathbf{0}$ for $i = -q+2, \dots, 0$.

2.3 Recursive estimation of EGARCH model

In the previous section, the specific form of the EGARCH model suitable for recursive estimation was introduced. One should remind that the conditional variance is assumed in the logarithmic form (5). The corresponding vector notation looks as follows:

$$\sigma_t^2(\boldsymbol{\theta}) = \exp(\boldsymbol{\varphi}_t^T(\boldsymbol{\theta})\boldsymbol{\theta}) \,, \tag{25}$$

where

$$\boldsymbol{\theta} = (\alpha_0, \alpha_1, \dots, \alpha_p, \delta_0, \dots, \delta_q, \gamma_0, \dots, \gamma_q)^T, \tag{26}$$

and $\boldsymbol{\varphi}_t(\boldsymbol{\theta})$ is such that (25) holds. The derivation provides the corresponding recursive algorithm in the form:

$$\hat{\boldsymbol{\theta}}_t = \hat{\boldsymbol{\theta}}_{t-1} + \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t(y_t^2 - \exp(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1}))}{(\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t)\exp(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_{t-1})} \,, \tag{27}$$

$$\hat{\mathbf{P}}_t = \frac{1}{\lambda_t}\left[\hat{\mathbf{P}}_{t-1} - \frac{\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}}{\hat{\boldsymbol{\psi}}_t^T\hat{\mathbf{P}}_{t-1}\hat{\boldsymbol{\psi}}_t + \lambda_t}\right], \tag{28}$$

$$\begin{aligned}\hat{\boldsymbol{\varphi}}_{t+1} = (1, \hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t, \dots, \hat{\boldsymbol{\varphi}}_{t+1-p}^T\hat{\boldsymbol{\theta}}_{t+1-p}, \frac{y_t}{\sqrt{\exp(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t)}}, \dots, \frac{y_{t-q}}{\sqrt{\exp(\hat{\boldsymbol{\varphi}}_{t-q}^T\hat{\boldsymbol{\theta}}_{t-q})}}, \\ \frac{|y_t|}{\sqrt{\exp(\hat{\boldsymbol{\varphi}}_t^T\hat{\boldsymbol{\theta}}_t)}} - \sqrt{2/\pi}, \dots, \frac{|y_{t-q}|}{\sqrt{\exp(\hat{\boldsymbol{\varphi}}_{t-q}^T\hat{\boldsymbol{\theta}}_{t-q})}} - \sqrt{2/\pi})^T,\end{aligned} \tag{29}$$

$$\hat{\boldsymbol{\psi}}_{t+1} = \hat{\boldsymbol{\varphi}}_{t+1} + \sum_{i=1}^{p}\hat{\alpha}_{i,t}\hat{\boldsymbol{\psi}}_{t+1-i} + \sum_{j=1}^{q}\frac{\hat{\delta}_{j,t}y_{t-j} + \hat{\gamma}_{j,t}|y_{t-j}|}{2\sqrt{\exp(\hat{\boldsymbol{\varphi}}_{t-j}^T\hat{\boldsymbol{\theta}}_{t-j})}}\hat{\boldsymbol{\psi}}_{t-j} \tag{30}$$

for $t \in \mathbb{N}$.

One can supplement the algorithm with the following initial estimates: $\hat{\boldsymbol{\theta}}_0 = (\frac{1}{n}\sum_{i=1}^{n} y_{1-i}^2[1-[p+2(q+1)]\eta], \eta, \dots, \eta, \eta, \dots, \eta, \eta, \dots, \eta)^T$, where η is a small positive constant satisfying $[p+2(q+1)]\eta < 1$ for a suitable n, $\hat{\mathbf{P}}_0 = c\mathbf{I}$, where c is a suitable constant, $\hat{\boldsymbol{\varphi}}_1 = (1, k, \dots, k, \frac{y_0}{\sqrt{\exp(k)}},$

$\dots, \frac{y_{-q}}{\sqrt{\exp(k)}}, \frac{|y_0|}{\sqrt{\exp(k)}} - \sqrt{2/\pi}, \dots, \frac{|y_{-q}|}{\sqrt{\exp(k)}} - \sqrt{2/\pi})^T$ with k equal to a small positive constant, $\hat{\boldsymbol{\psi}}_1 = \hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\psi}}_i = \mathbf{0}$ for $i = \min\{-p+2, -q+1\}, \dots, 0$.

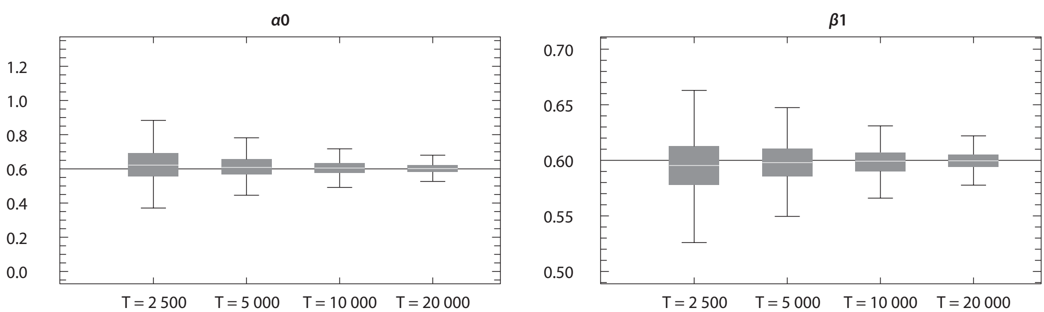
Remark: Other models were considered, e.g., the matrix extension of GARCH model respecting the interactions of model components. However, due to an extensive number of parameters, the numerical outputs (mainly volatility predictions) were not satisfactory. It is a well-known fact that the quality of GARCH modeling decreases with increasing number of model parameters. For the same reason, only the lowest orders of models were applied numerically in this paper (mostly $p = q = 1$). The identification criteria (mainly AIC) mostly confirmed that such order choices do not differ significantly from the optimal ones. As the estimation of μ_t is concerned, in the context of high-frequency financial data its approximation by zero level is realistic. Other alternatives consist in the application of various econometric methods (see, e.g., Cipra, 2020). Finally, the impact of distribution of residuals ε_t's is covered approximatively by using the quasi log-likelihood approach.

2.4 Simulation study

An extensive simulation study was performed to evaluate the proposed recursive algorithms. In particular cases 1 000 time series of length 20 060 were simulated for particular models applying $\mu_t = 0$ and $\varepsilon_t \sim iid\ \mathrm{N}(0,1)$ in (1). The first 60 of the 20 060 observations were used in order to determine the initial estimates, as suggested in previous sub-sections. The remaining 20 000 observations are used for the subsequent on-line estimation.

To compare recursive algorithms, figures with boxplots were produced for each model. In this subsection we present only the case of IGARCH(1,1) model (3) with parameters $\alpha_0 = \beta_1 = 0.6$ (hence $\alpha_1 = 0.4$), see Figure 1. For each parameter, boxplots of estimates at times $T = 2\,500$, $T = 5\,000$, $T = 10\,000$ and $T = 20\,000$ are shown. Every box shows the range between the first and third quartile of obtained estimates, the white bar represents the median and the long line indicates the true value of the parameter.

Figure 1 Boxplots of the IGARCH(1,1) process estimates with $(\alpha_0, \beta_1) = (0.6, 0.6)$



Source: Own construction

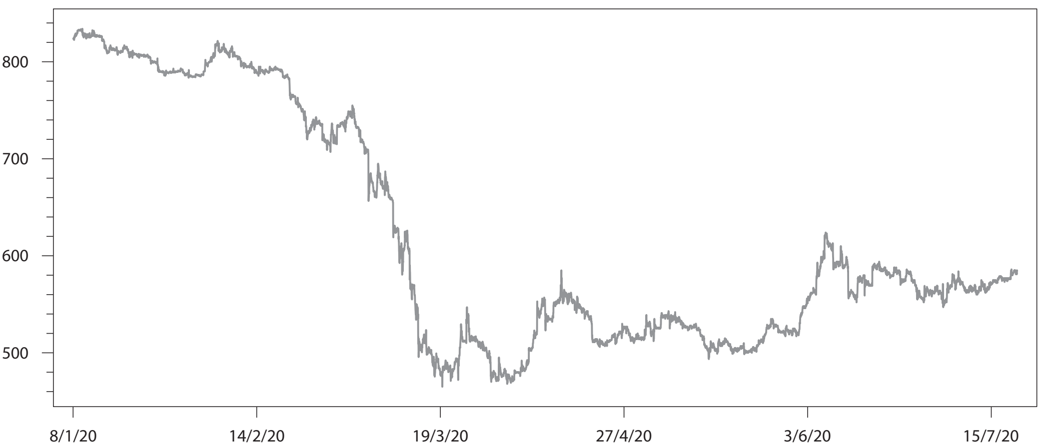
It can happen (particularly, when the true parameters are close to the borders of corresponding parameter constraints, e.g., for IGARCH(1,1) with $\alpha_0 = 0.2$ and $\beta_1 = 0.9$ that the convergence of recursive algorithms is slower when some parameters are overestimated and remaining parameters underestimated with a mutual compensation effect. Fortunately, such behavior does not distort the volatility estimation being the target output of particular recursive algorithms.

3 REAL DATA EXAMPLES

In the previous section, the ability of models to estimate parameters was verified. The primary role of the proposed recursive algorithms is their use for modeling high frequency time series. For example, for stock traded assets, one can encounter one-minute and even tick data. With such a high frequency, the volume of data is great even in a short period. In this case study, the given algorithms will be applied to a real-time series and will be also compared mutually in order to select the most suitable model for the observed time series. Even more important than the estimated parameters in a given model is the estimation of volatility, which plays a key role when trading the given asset.

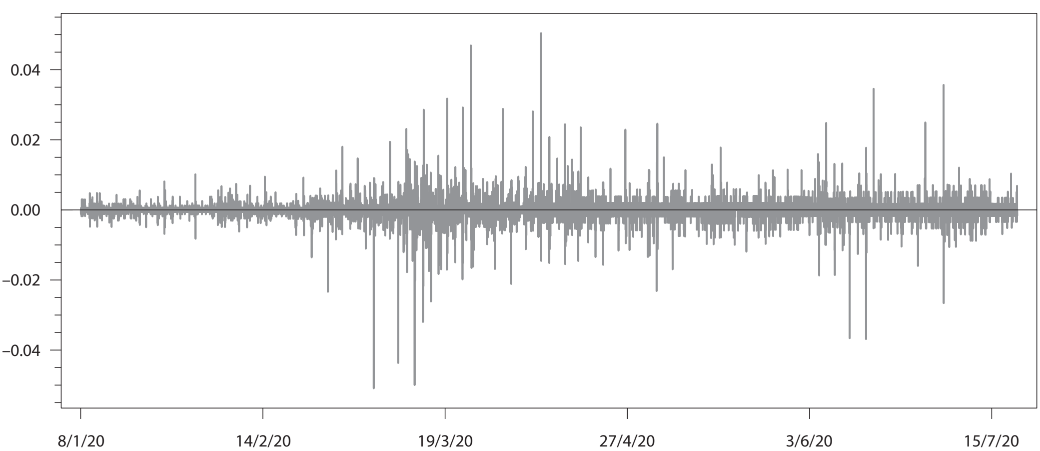
The time series consists of the stock prices of the company Komerční banka (KB) from January to July 2020. The benefit of this choice is the fact that KB operates on the Czech capital market, its stocks

Figure 2 Komerční banka stock prices from January 8th, 2020 to July 22nd, 2020, five-minute data



Source: Bloomberg

Figure 3 Logarithmic returns of Komerční banka from January 8th, 2020 to July 22nd, 2020, five-minute data



Source: Own construction

are traded on the Prague Stock Exchange (PX), and therefore, the data from Czech environment are used in the study. The second advantage consists in the investigated data period. In the given time interval, the coronavirus epidemic started in the Czech Republic, which significantly affected stock prices. Thus, we can verify how the algorithms cope with possible crises.

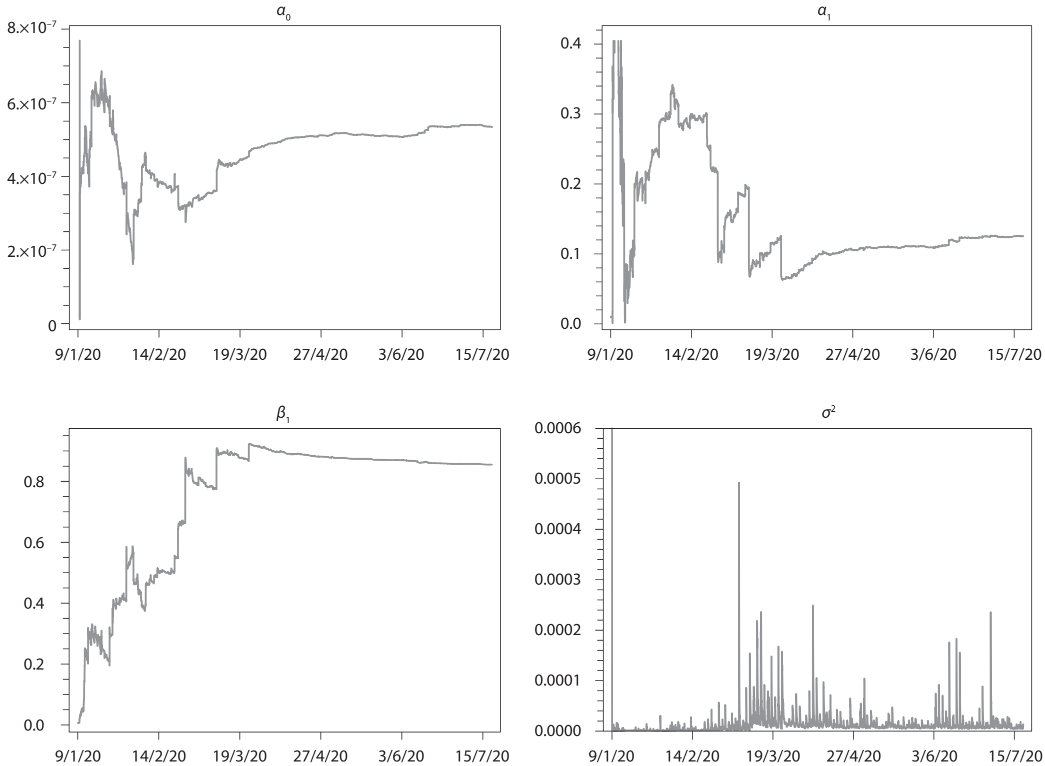
Five-minute data for the period from January 8^{th}, 2020 to July 22^{nd}, 2020 are available using the Bloomberg database (10 282 observations in total). These stock prices are plotted in Figure 2.

In practice, logarithmic returns are usually considered for modeling, the aim of which is, among other consequences, to make the time series stationary. Generally, logarithmic returns r_t are calculated as $r_t = \ln \frac{P_t}{P_{t-1}}$, where P_t and P_{t-1} are prices of an asset at times t and $t - 1$ (in our case stock prices). The time series of the logarithmic returns of KB in the given time period is shown in Figure 3.

3.1 Estimation

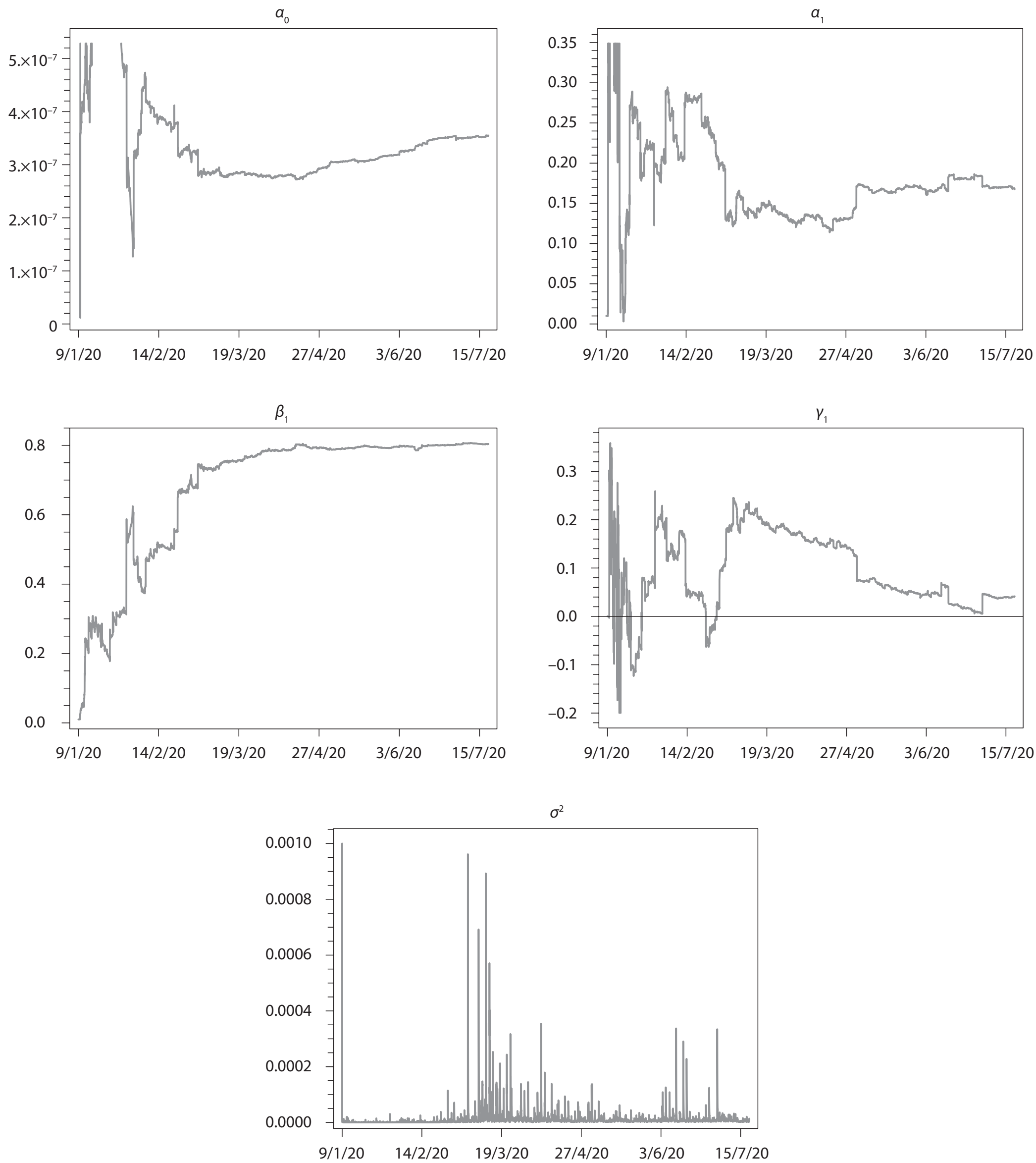
We can proceed now to the estimation of the presented models ($p = 1$, $q = 1$) for the given time series of the logarithmic returns. For each model, the graphs of the development of parameters over time and the estimate of the conditional variance are given.

Figure 4 KB: Parameter estimates and conditional volatility – GARCH(1,1)



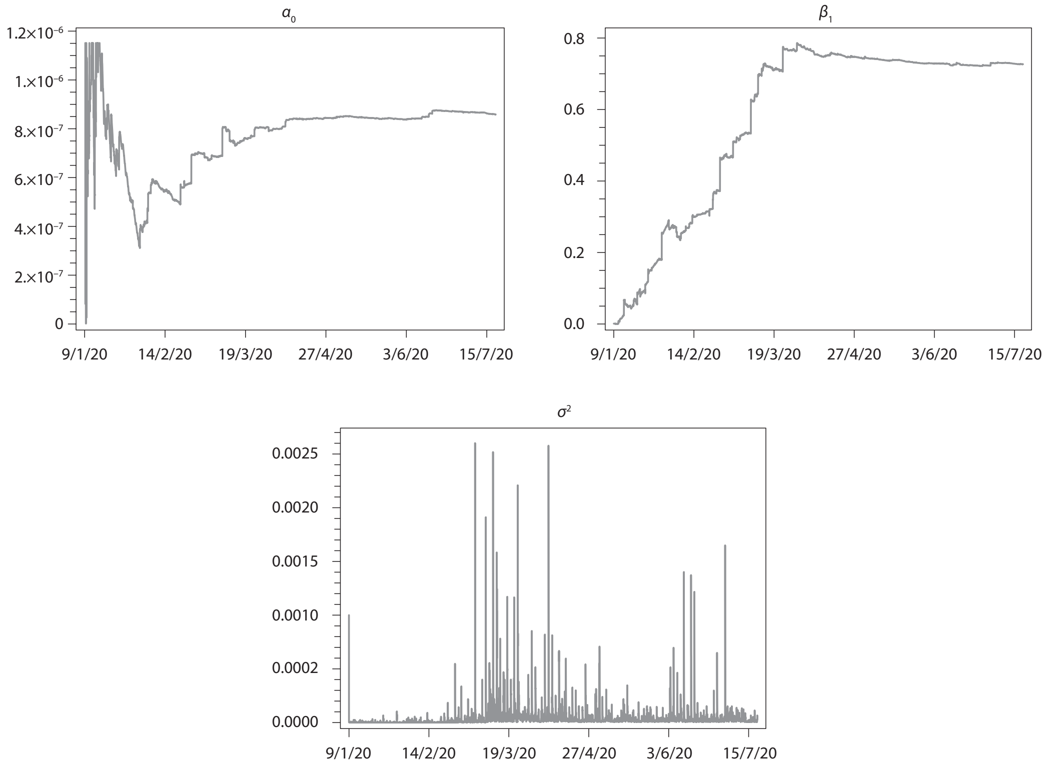
Source: Own construction

Figure 5 KB: Parameter estimates and conditional volatility – GJR-GARCH(1,1)



Source: Own construction

Figure 6 KB: Parameter estimates and conditional volatility – IGARCH(1,1)



Source: Own construction

Figure 7 KB: Parameter estimates and conditional volatility – EGARCH(1,1)

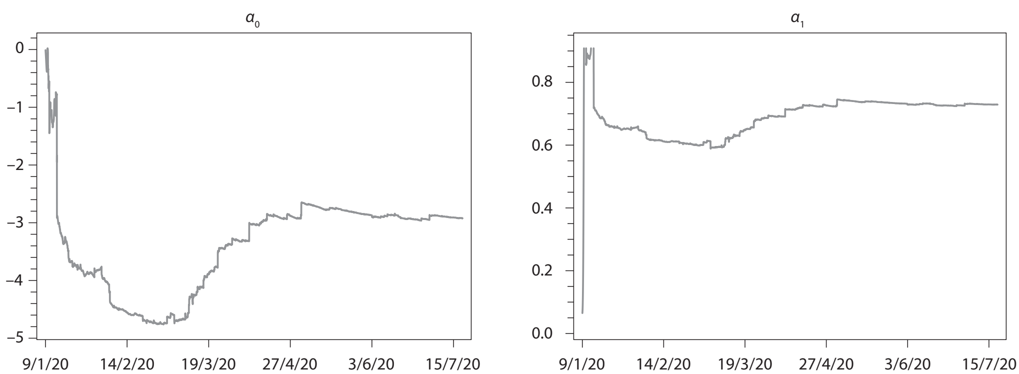
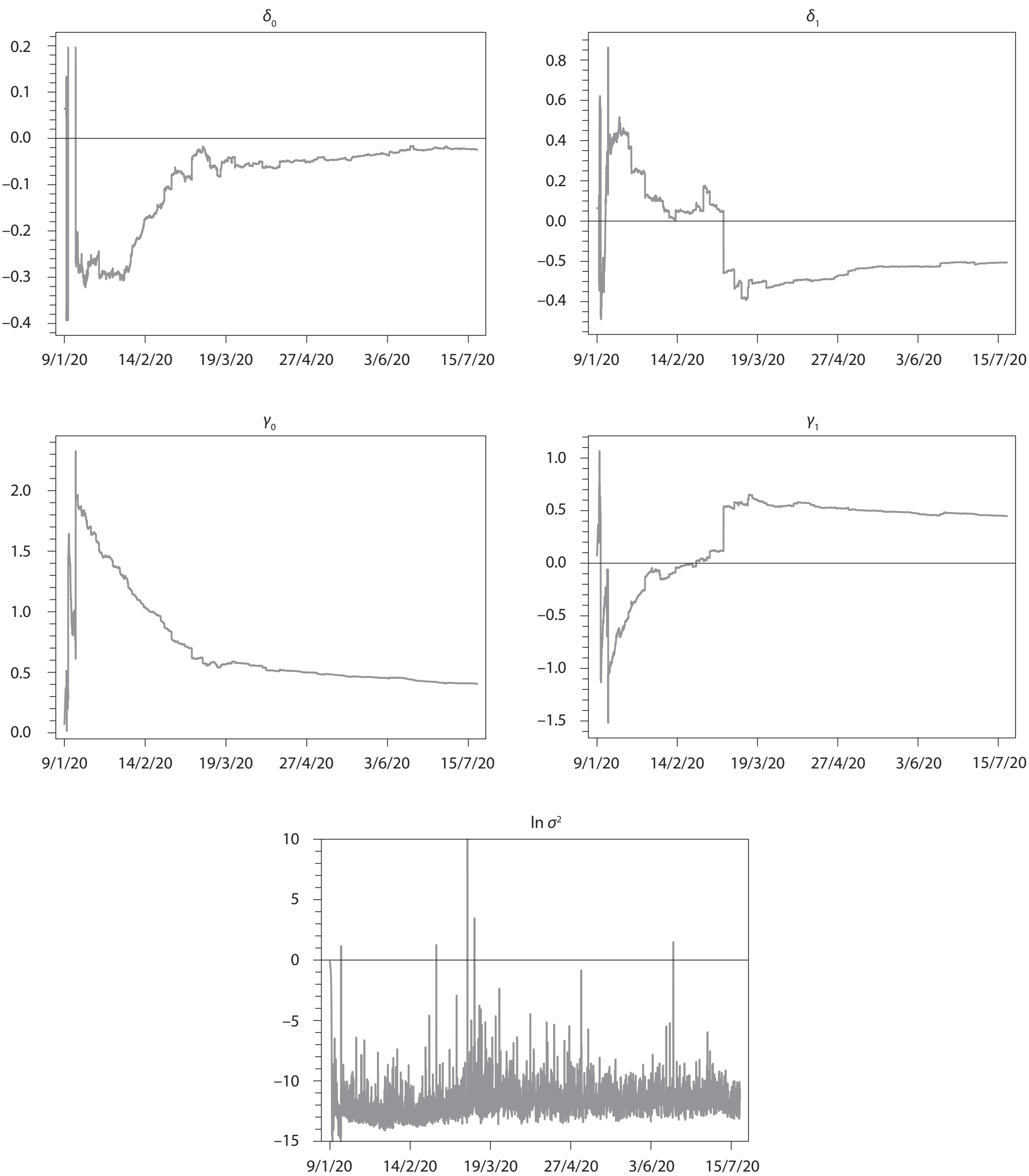


Figure 7 (continuation)



Source: Own construction

For all models except for the EGARCH model, one can observe a similar shape of the conditional volatility graph. The graphs differ only in the scale. In the case of the EGARCH model, the logarithm of the conditional variance is presented which is the output of the model.

The onset of the coronavirus crisis can be clearly identified in Figures 4–7. Significant changes in parameters are visible in this period. Moreover, a considerable increase is also evident in the estimated volatility, which was caused by a significant drop in KB stock prices. The second period of increased volatility occurred in June 2020, when the stock price gradually increased. For the GJR-GARCH and the EGARCH models taking into account the leverage effect, it is possible to verify that the given financial time series really has this characteristic. In the GJR-GARCH(1,1) model, the leverage effect is indicated by positive values of the parameter γ_1. In the EGARCH(1,1) model, the leverage effect is present when the parameters δ_0 and δ_1 are negative (the significance of positive or negative values of estimated parameters gamma and delta can be tested statistically). In both models, this is true for major parts of the time series.

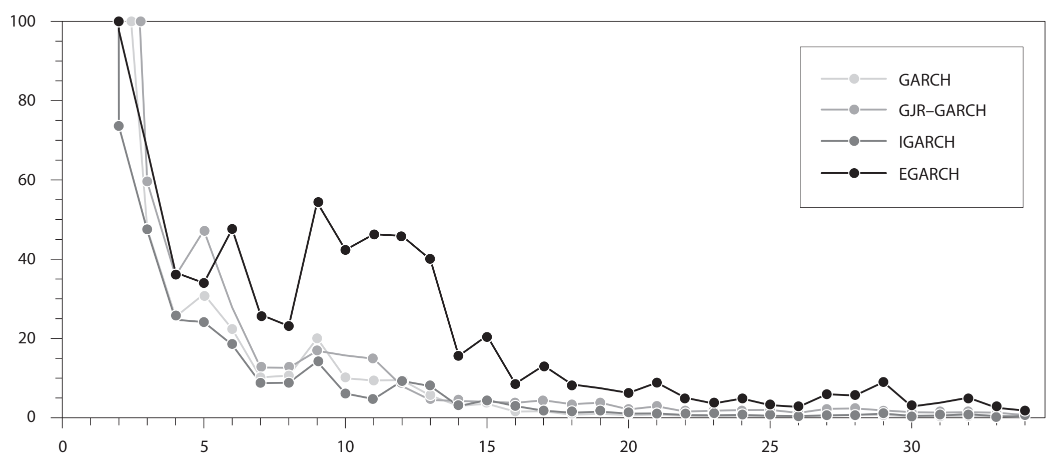
3.2 Risk prediction

Since in the case of financial time series, the ability of risk prediction is very important, we decided to use the measure for the accuracy of volatility predictions by particular models, which is inspired by MAPE (Mean Absolute Percentage Error). To decide on the best model in different periods, we divided the time series into segments with length of three hundred observations, and the following percentage quantities were calculated for each segment:

$$\widetilde{MAPE}_{i+1} = \frac{100}{300} \sum_{t=i*300+1}^{(i+1)*300} \left| \frac{\hat{\sigma}^2_{t+1}(t+1) - \hat{\sigma}^2_{t+1}(t)}{\hat{\sigma}^2_{t+1}(t+1)} \right|, \tag{31}$$

where i is taken as $i = 0, \ldots, 33$, $\hat{\sigma}^2_{t+1}(t+1)$ is the estimated volatility at time $t + 1$ and $\hat{\sigma}^2_{t+1}(t)$ is the one-step ahead prediction of the conditional variance value at time $t + 1$ with the information available till time t. Thus, the proposed measure assesses, how the given model predicts volatility one step ahead. The lower the value, the better the predictions are. The advantage of this approach is that one can model the given time series using more models parallelly and choose the best model on-line. Figure 8 and Table 1 show a comparison of $\widetilde{MAPE}$ for the given time series and the particular segments, as introduced above.

Figure 8 KB: Graph of computed $\widetilde{MAPE}$ (in percentage)



Source: Own construction

Table 1 KB: Computed $\widetilde{MAPE}$ (in percentage)

	2	5	10	15	20	25	30	34
GARCH	139.84	31.18	9.97	3.53	0.61	0.40	0.36	0.22
GJR-GARCH	211.87	47.28	15.72	4.03	2.11	1.98	1.40	0.68
IGARCH	73.52	24.14	6.12	4.47	1.05	0.51	0.48	0.35
EGARCH	97.67	33.99	42.34	20.71	6.26	3.18	2.84	1.81

Source: Own construction

Predictions are not very accurate at the beginning of the time series. However, this can be expected due to the initial calibration of the models. Later, the forecasts noticeably improved. One can notice that for the GARCH(1,1), the GJR-GARCH(1,1) and the IGARCH(1,1) models, $\widetilde{MAPE}$ decreased faster. This is could be explained by a higher number of parameters in the EGARCH(1,1) model, which takes a longer time to calibrate itself. However, in the second half of the time series, the values of $\widetilde{MAPE}$ are already very similar for all models. Figure 8 also shows that the IGARCH(1,1) model was the best in the first third of the time series, while the GARCH(1,1) model was the best in the rest. The conclusions of the residual analysis, e.g., by calculating the AIC, correspond to the findings obtained from the $\widetilde{MAPE}$ comparison of the particular models.

Figure 9 KB: Realized daily volatility calculated by intra-day data and daily volatility predictions using particular models

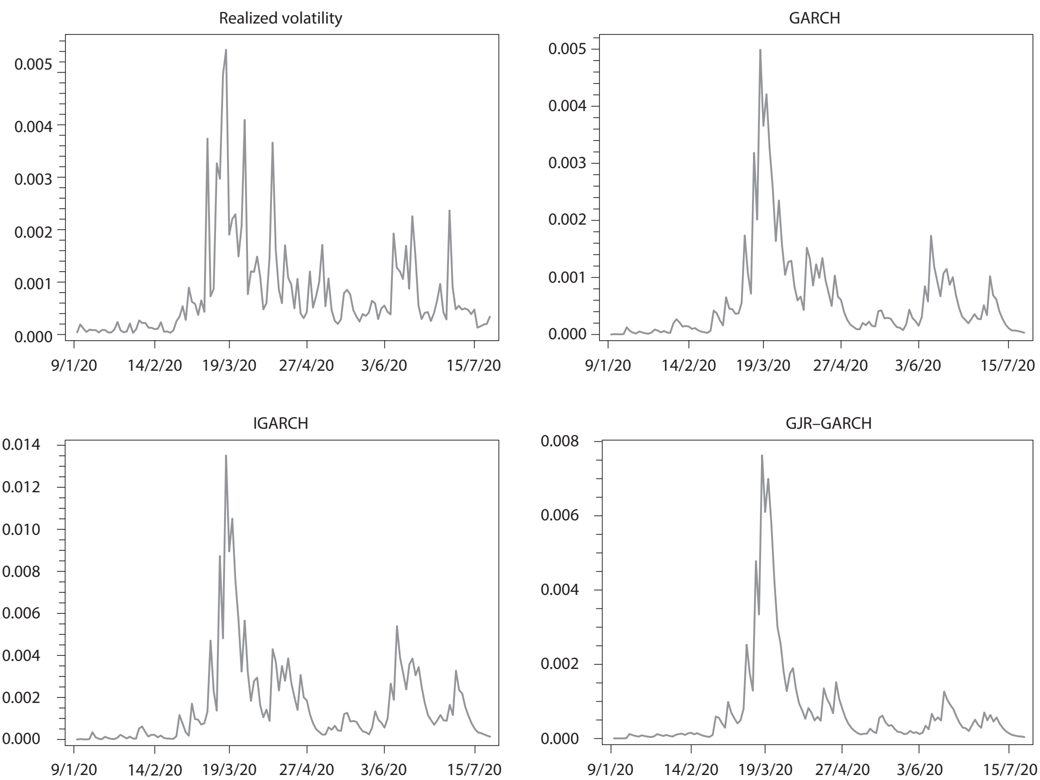
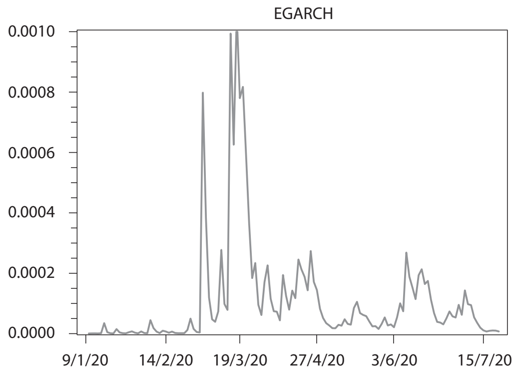


Figure 9 (continuation)



Source: Own construction

Since the five-minute predictions need not be relevant for financial practice, we have tried to evaluate the risk prediction potential of the recursive methodology also by using daily data which enables to compare the outputs of particular models with the realized volatility calculated by means of the original intra-day data (see, e.g., Patton and Sheppard, 2009). For instance, Figure 9 plots the realized daily volatilities calculated by intra-day data and the corresponding daily volatility predictions provided by particular models which are estimated recursively using daily data in particular models (for the same stocks, the same period and the same model orders $p = 1$ and $q = 1$). The consequent analysis shows that the outputs by the model GARCH(1,1) are closest to the realized volatility.

Other datasets have been used to verify the behavior of recursive estimates, e.g., the ČEZ stock prices from January to July 2020. As with KB, the stock prices were strongly influenced by the pandemic.

CONCLUSION

This article focuses on recursive algorithms for GARCH model modifications and their use for on-line estimation. The main advantages of the recursive estimation in the context of high-frequency time series are low memory requirements and overall speed. Thus, the proposed recursive algorithms can be applied to financial time series, which are typical representatives of high-frequency time series. In addition to the survey of recursive algorithms for GARCH model modifications, a nonnegligible benefit of this article consists in the numerical case study. A high-frequency time series of logarithmic returns of Komerční banka was investigated using the recursive algorithms from Section 2. The considered data are also of interest due to the fact that the observations are recorded in the period when the coronavirus pandemic started in the Czech Republic. The presented outputs certify that the algorithms have clearly identified the pandemic. Finally, we suggested an efficient methodology to compare recursive risk predictions among particular models.

ACKNOWLEDGEMENT

The paper is supported by the Czech Science Foundation 19-28231X. The authors thank to Dr. Radek Hendrych for an efficient help when preparing this paper and to anonymous reviewers for helpful comments and improvements.

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